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TREATISE

ON

O P T I C S.

BY

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The student is recommended in his first reading of this subject to omit the following articles :

59, 60, 80, 81, 93, 102, 107, 108, 110, 111, 113, 115, 121, 125—131, 186—190,
193, 208, 210, 224—226, 231, 232, 235, 249.

OPTICS.

SECTION I.

GENERAL PROPERTIES OF LIGHT, AND DIRECT REFLECTION AND REFRACTION.

1. WHEN a material object is presented before us, we become by vision sensible of its existence and figure. In such a case light is said to be propagated from the object to our eyes, and the science of Optics has for its design the examination of the circumstances of such propagation.

The science is divided into Geometrical and Physical Optics. In Geometrical Optics the circumstances of the transmission and modification of light are computed on certain laws established by experiment; in Physical Optics these laws are accounted for on hypotheses of the structure of bodies, and of the matter filling the space in which they are placed. In a similar manner in Geometrical Astronomy the phenomena of heavenly bodies are calculated on observed laws which their apparent motions are found to obey; in Physical Astronomy these apparent laws are shewn to result from the hypothesis of Gravitation.

The former branch of the science is the subject of the present treatise, wherein from certain laws established by experiment under simple circumstances, the course of light under more complex circumstances is computed, and the results applied to the construction of Optical Instruments. These investigations will be conducted in independence of

the Physical branch of the subject, since the experimental laws on which we commence are equally true, whatever be the correctness of the hypothesis which professes to account for them.

2. DEF. A body is called *self-luminous* when it is capable in itself of making our eyes sensible of its existence.

Other bodies are luminous by reflection; that is, they require the presence of another luminous body to render them visible.

OBS. When an origin of light is mentioned, it is always to be considered a mathematical point.

3. DEF. Whatever allows the transmission of light is called a *medium*.

4. DEF. The smallest portion of light which can be separately transmitted, stopped, or reflected, is called a *ray*.

In an uniform medium it will be assumed that the course of a ray of light is a straight line; this law experiment shews to be in general true with certain cases of apparent exception.

5. DEF. An assemblage of rays proceeding from a luminous point, is a *pencil of light*.

A pencil of light, unless the contrary be expressed, is considered to be in form a cone with a circular directrix, having the origin of light for the vertex, and when the origin is infinitely distant, this cone has a circular cylinder as its limiting form. The geometrical axis of this cone or cylinder is called the axis of the pencil.

If the rays of a pencil of light produced in a contrary direction to that of propagation meet in a point, the pencil is divergent; if the origin be infinitely distant, its limiting form is a pencil of parallel rays; if the directions of the rays produced in direction of propagation meet in a point, the pencil is convergent.

The degree of divergence or convergence is measured by the semi-vertical angle of the cone which the pencil forms.

6. DEF. When a pencil meets the surface of any substance, the incidence is called direct if the axis of the pencil coincide with the normal to the surface at the point of incidence; in other cases the incidence is called oblique.

7. When light is incident on the surface of a medium different from that in which it is proceeding, a portion is dispersed and makes the surface visible, another portion is in general reflected in the latter medium according to a law to be stated, and in certain cases a third portion enters the new medium according to another law and is said to be refracted. The course of the reflected and refracted rays, where each exists, may be separately considered.

The following are the two experimental laws to which reference has been made.

8. *Law of Reflection.*

When a ray is reflected at the surface of a medium,

(1) the incident and reflected rays lie in the same plane with the normal to the surface at the point of incidence, and on opposite sides of it,

(2) the angles which the incident and reflected rays make with the normal to the surface at the point of incidence are equal.

9. *Law of Refraction.*

When a ray is refracted at the surface of a medium,

(1) the incident and refracted rays lie in the same plane with the normal to the surface at the point of incidence, and on opposite sides of it;

(2) the sines of the angles which the incident and refracted rays make with the normal to the surface at the point of incidence, have a ratio depending only on the media between which the refraction takes place, and the nature of the light.

The sine of the angle of incidence divided by the sine of the angle of refraction is denoted by the quantity μ , which is constant therefore at whatever angle of incidence a ray of the same kind of light is refracted from one given medium into another.

This quantity is called the refractive index between the two media for that species of light. It is a parameter admitting of variation,

- (1) if the nature of the light be altered;
- (2) if the relation between the two media be altered.

It will in general be supposed that the refraction takes place into a denser medium, in which case μ is greater than 1, and the angle of refraction less than the angle of incidence.

10. The process by which these and physical laws in general are established experimentally is this. Direct experiments render the law probable; these however are seldom of such minute accuracy as to prove the law exactly true: next on supposition of the truth of the law in question, the circumstances of more complex phenomena are computed, and when the results of these computations are found in repeated instances of various kinds minutely to agree with observations, we have a very high degree of probability of the strict truth of the law. It is important to observe that of the laws of physical science we have only moral certainty, a certainty arising only from the improbability that an untrue principle should happen successfully to explain a great variety of phenomena. The first law of motion in Dynamics, for example, is made probable by experiments on bodies on the earth; it is proved by the agreement of the motion of the heavenly bodies, calculated on supposition of its truth, with the motions which they are observed to have. On such foundations all the laws of natural philosophy rest.

11. Light is propagated with finite velocity.

The eclipses of Jupiter's satellites are observed to happen sooner, when he is in geocentric opposition, and consequently

nearest to the Earth, and later, when he is in conjunction or farthest from the Earth, than they ought according to calculations made on supposition that he is at his mean distance from the Earth. The difference is accounted for by the hypothesis of light requiring a finite time for its transmission. The supposition is confirmed by its satisfactorily explaining the apparent displacements of the heavenly bodies called Aberration.

The coefficient of aberration being the same for heavenly bodies at different distances, it appears that the velocity of light is uniform. From such observations as have been mentioned, this velocity in vacuo is found to be 192500 miles per second.

12. *Illumination of surfaces.*

When a pencil of light emanates from a luminous point and is propagated in an uniform medium, if we suppose its intensity unaltered by the absorption of any portion of it by the medium, yet from other causes the illumination at any point of a surface exposed to the light is different in different positions and at different distances of the surface.

13. DEF. The illumination at any point of a surface exposed to light is measured by the quantity I , where $I\kappa$ is the illumination of an indefinitely small area κ of the surface contiguous to the point in question, referred to some standard degree of illumination as an unit.

Hence if the same quantity of light fall on two very small areas, $I\kappa$ being the same for each, the illuminations at any point of these areas are inversely as the areas.

14. When a small plane area is illuminated by a pencil of rays emanating from a point,

$$\text{illumination at any point} \propto \frac{\text{cosine of angle of incidence}}{\text{square of distance from origin}}.$$

Let QAB (fig. 1) be a small conical pencil of light from an origin Q ; ACB , aCb a circular and oblique section of

it through a point C in the axis. If aCb be a small plane area illuminated by the pencil,

(1) In all sections parallel to aCb the quantity of light in the pencil being the same, the illumination at a point is inversely as the area of the section (13), i. e. inversely as $(\text{dist.})^2$ from Q .

(2) In all sections through C at different inclinations to the axis, the quantity of light received being the same as that received by ACB , the illumination \propto area inversely. But if the pencil be supposed so small that ACB may be regarded as the orthogonal projection of aCb

$$\text{area } aCb = \frac{\text{area } ACB}{\cos ACa} \quad (\text{Hymers' Anal. Geom. 81}).$$

$$\therefore \text{illumination in } aCb = \text{illumination in } ACB \times \cos ACa, \\ \propto \cos ACa,$$

and ACa being the inclination of the planes ACB , aCb is the angle between the perpendiculars to these planes at C , or is the angle of incidence of QC .

Hence when both the distance and angle of incidence vary together,

$$\text{illumination at any point of the area} \propto \frac{\text{cosine of angle of incidence}}{(\text{distance})^2}.$$

COR. Illumination at any point of the area

$$= C \frac{\text{cosine of angle of incidence}}{(\text{distance})^2};$$

where C depends only on the brightness of the illuminating point and is the illumination at any point of a small area directly exposed to the pencil at a distance = 1 from the origin.

Direct Reflection and Refraction.

15. DEF. Let Q (fig. 3) be the origin of a pencil of rays whose axis QA is incident directly at A on a plane or

spherical reflecting or refracting surface. Then QA is the axis of the reflected or refracted pencil. Let QR be any ray incident at R whose direction after reflection or refraction cuts the axis in q , since the normal at R lies in the plane QAR . Then as R is taken nearer to A , the point q will approach some point F in the axis as its limiting position, and the distance Fq may be made less than any assignable distance. This limiting position of the point q is called the Geometrical focus of the reflected or refracted pencil.

16. DEF. The principal focus of a spherical reflecting or refracting surface is the Geometrical focus of a pencil of parallel rays incident directly upon the surface parallel to a fixed diameter called the axis of the surface.

17. DEF. The focal length of a spherical reflecting or refracting surface is the distance between the surface and the principal focus.

18. A pencil of parallel rays consists of parallel rays after reflection at a plane surface.

Let $QR, Q'R'$ (fig. 4) be any two rays of a pencil of parallel rays incident on a plane reflecting surface at the points R, R' . Let RS be the direction of the ray QR after reflection. Draw $RN, R'N'$ perpendicular to the surface at the points R, R' , and let $R'S'$ be the intersection of the plane SRR' with the plane $Q'R'N'$.

Now $\left. \begin{matrix} QR \\ RN \end{matrix} \right\}$ are parallel to $\left\{ \begin{matrix} Q'R' \\ R'N' \end{matrix} \right.$;

\therefore the plane QRS is parallel to the plane $Q'R'S'$ (Euc. xi. 15);

\therefore the st. line RS st. line $R'S'$ (Euc. xi. 16).

Also because $\left. \begin{matrix} QR \\ NR \end{matrix} \right\}$ are parallel to $\left\{ \begin{matrix} Q'R' \\ R'N' \end{matrix} \right.$;

$\therefore \angle QRN = \angle Q'R'N'$ (Euc. xi. 10).

Similarly $\angle SRN = \angle S'R'N'$;

But $\angle QRN = \angle SRN$;

$\therefore \angle Q'R'N' = \angle S'R'N'$,

and $R'S'$ lies in the plane $Q'R'N'$; therefore it is the direction of the ray $Q'R'$ after reflection, and it has been proved to be parallel to RS . But QR , $Q'R'$ are any two rays of the incident pencil; therefore the reflected pencil consists of parallel rays.

19. A pencil of parallel rays consists of parallel rays after refraction at a plane surface.

Let QR , $Q'R'$ (fig. 5) be any two rays of a pencil of parallel rays incident on a plane refracting surface at the points R , R' . Let SR be the direction of the ray QR after refraction. Draw RN , $R'N'$ perpendicular to the surface at the points R , R' , and let $S'R'$ be the intersection of the plane SRR' with the plane $Q'R'N'$.

Now $\left. \begin{matrix} QR \\ RN \end{matrix} \right\}$ are parallel to $\left. \begin{matrix} Q'R' \\ R'N' \end{matrix} \right\}$;

\therefore the plane QRS is parallel to the plane $Q'R'S'$;

\therefore the st. line SR st. line $S'R'$.

Also because $\left. \begin{matrix} QR \\ RN \end{matrix} \right\}$ are parallel to $\left. \begin{matrix} Q'R' \\ R'N' \end{matrix} \right\}$;

$\therefore \angle QRN = \angle Q'R'N'$.

Similarly $\angle SRN = \angle S'R'N'$,

$$\begin{aligned} \therefore \sin Q'R'N' &= \sin QRN \\ &= \mu \cdot \sin SRN \\ &= \mu \cdot \sin S'R'N', \end{aligned}$$

and $S'R'$ lies in the plane $Q'R'N'$; therefore it is the direction of the ray $Q'R'$ after refraction, and it has been proved to be parallel to RS . But QR , $Q'R'$ are any rays of the incident pencil; therefore the refracted pencil consists of parallel rays.

20. If a pencil be incident directly upon a plane reflecting surface, to find its form after reflection.

Let Q (fig. 6) be an origin of light from which a pencil whose axis is QA is incident directly at A on a plane reflecting surface, QR any ray of the pencil incident at R and reflected in direction RS in a plane through QR and RN the perpendicular to the surface at R . Since by the law of reflection SR , QA lie in one plane, let them be produced to meet in q . Then since RN , qQ are parallel,

$$\begin{aligned} RQA &= QRN \\ &= SRN \text{ (by the law of reflection)} \\ &= RqA; \end{aligned}$$

\therefore the angles $\left. \begin{matrix} RqA \\ RAq \end{matrix} \right\}$ are equal to the angles $\left\{ \begin{matrix} RQA \\ RAQ \end{matrix} \right.$ each to each, and RA is common to the two triangles;

$$\therefore Aq = AQ.$$

Now QR is any ray of the incident pencil; therefore the directions of all the rays of the reflected pencil pass through the point q .

Hence the form of the reflected pencil is that of a cone whose axis is AQ and vertex the point q , equidistant with Q from the reflecting surface, and on the opposite side of it.

COR. 1. Since the angles RqA , RQA are equal, the divergence of the incident and reflected pencils is the same. (5).

COR. 2. If the incident pencil be convergent, a similar investigation will shew that it will converge after reflection to a point equidistant from the surface with the point of convergence of the incident pencil and on the opposite side of it, the degree of convergence being unaltered.

COR. 3. The ray QR after reflection cuts the axis in q , and the same is true in the limit when R moves up to A ; therefore q is the geometrical focus of the reflected pencil. (15).

21. The succeeding cases of direct reflection and refraction have greater difficulty than the last investigation, because in none of them will the pencil after reflection or refraction accurately pass through a point. Our attention will at present be confined to plane and to spherical surfaces.

A pencil whose origin is Q (fig. 7) and axis QA incident directly on a plane or spherical reflecting or refracting surface may be considered composed of a series of conical surfaces of rays, as QRr , with a common vertex Q and common axis QA . The rays of this conical surface will all be reflected or refracted similarly with respect to QA , and therefore their directions after reflection or refraction will form another conical surface with vertex q and axis QA . Thus the reflected or refracted pencil will consist of a series of conical surfaces with a common axis, but different vertices. The limiting position of the vertex q is the geometrical focus of the pencil. (15).

The calculation of the form of a direct pencil after reflection or refraction will consist of two parts:

- (1) the determination of the geometrical focus;
- (2) the determination of the vertex q of any one of the cones of rays above described.

The equation which gives the distance Aq will not in general admit of direct solution, and is solved by successive approximations, the approximation being conducted according to powers of AR , the half breadth of the conical shell in question, which in such pencils as occur in the computations of instruments is very small compared with the other lines involved in the equation. The square and higher powers of AR being at first neglected, a first approximate value of Aq is obtained. Next, by neglecting the cube and higher powers of AR , and substituting in the coefficient of AR^2 the approximate value of Aq before determined, a second approximate value is obtained. By this means the value of Aq may be determined to any degree of accuracy. It is found sufficient to carry the calculation as far as AR^2 .

22. The following proposition is intended to shew that, as has been just mentioned, an approximate value of a quantity to be determined may be substituted in the small terms of the equation.

Suppose v a quantity whose value is to be found, and which is given implicitly by the equation

$$v = V + f(v) \cdot y^2 \dots \dots \dots (A)$$

where V involves known quantities only, and is independent of y , the small quantity by powers of which the approximation is conducted, and $f(v)$ is a function of v and known quantities.

Then by substitution

$$f(v) = f\{V + f(v) \cdot y^2\} = f(V) + f'(V) \cdot f(v) \cdot y^2 + \&c.$$

$$\therefore v = V + f(V) y^2 + f'(V) \cdot f(v) \cdot y^4 \dots$$

$$= V + f(V) \cdot y^2,$$

if the approximation extend only to the square of y .

It appears therefore that in the term of (A), which involves y^2 , we may substitute the value of v obtained by neglecting y^2 , and the result will be true to the order of y^2 .

23. To find the geometrical focus of a pencil, after direct refraction, at a plane surface.

Let Q (fig. 8) be the origin of a pencil whose axis QA is incident directly at A on a plane refracting surface, then QA is the axis of the refracted pencil. Let QR be any ray incident at R and refracted in a direction which cuts the axis in q . Let F be the geometrical focus or limiting position of q .

Let $AQ = u$, $AF = v$, lines being considered positive when measured from A in a direction contrary to that of the incident light.

Now RQA , RqA being equal to the angles of incidence and refraction of the ray QR ,

$$\sin RQA = \mu \sin RqA,$$

$$\frac{AR}{RQ} = \mu \cdot \frac{AR}{Rq},$$

$$Rq = \mu \cdot RQ.$$

In the limit when R moves up to A and q to F ,

$$AF = \mu \cdot AQ, \text{ or } v = \mu u,$$

which gives the position of the geometrical focus of the refracted pencil.

Obs. In this proposition, as will be done in other cases, the pencil is considered divergent, but attention to the sign of u will make the result applicable to a convergent pencil. In such a case u is negative, and $v = \mu u$ indicates that the geometrical focus lies in a negative direction, that is, behind the surface at a distance from it determined by the numerical value of μu .

24. When a pencil is incident directly on a plane refracting surface, to find the point where the direction of a given ray after refraction cuts the axis.

Let Q (fig. 9) be the origin of a pencil whose axis QA is incident directly at A on a plane refracting surface, then QA is the axis of the refracted pencil. Let QR be any ray incident at R , and refracted in a direction which cuts the axis in q , the position of which is to be determined.

Let $AQ = u$, $Aq = v'$; lines being considered positive when measured from A in a direction contrary to that of the incident pencil. Also let $AR = y$, a quantity of which the cube and higher powers may be neglected.

Now RQA , RqA being equal to the angles of incidence and refraction of the ray QR ,

$$\sin RQA = \mu \sin RqA,$$

$$Rq = \mu \cdot RQ,$$

$$\begin{aligned}\sqrt{v'^2 + y^2} &= \mu \sqrt{u^2 + y^2}, \\ v' \left\{ 1 + \frac{y^2}{2v'^2} \right\} &= \mu u \left\{ 1 + \frac{y^2}{2u^2} \right\}, \\ v' &= \mu u + \left(\frac{\mu}{u} - \frac{1}{v'} \right) \frac{y^2}{2}.\end{aligned}$$

In the coefficient of y^2 we may substitute μu or v the first approximate value of v' , and the resulting equation will be true to the order of y^2 (22);

$$\begin{aligned}\therefore v' &= \mu u + \left(\frac{\mu}{u} - \frac{1}{\mu u} \right) \frac{y^2}{2}; \\ &= \mu u + \frac{\mu^2 - 1}{\mu} \cdot \frac{y^2}{2u}.\end{aligned}$$

COR. Hence it appears that the geometrical focus is the approximate position of the point q , when powers of y above the first are neglected.

25. DEF. When a pencil is directly reflected or refracted at a surface, the aberration of any ray is the distance between the geometrical focus and the point where the direction of that ray after reflection or refraction cuts the axis.

The aberration of a pencil is the aberration of the extreme ray in any section of the pencil through its axis.

Hence the aberration of the ray qR

$$\begin{aligned}&= v' - \mu u, \\ &= \frac{\mu^2 - 1}{\mu} \cdot \frac{y^2}{2u}.\end{aligned}$$

In this case the aberration is always from the refracting surface.

26. To find the geometrical focus of a pencil of rays after direct reflection at a spherical surface.

Let Q (fig. 10) be the origin of a pencil of light whose axis QA is incident directly on a spherical reflecting surface

of which O is the center; then AQ is the axis of the reflected pencil. Let QR be any ray incident at R and reflected in direction Rq , cutting the axis in q , F the geometrical focus. Join OR .

Let $AQ = u$, $AO = r$, $AF = v$, lines being considered positive, when measured from A in a direction contrary to that of the incident pencil.

$$\text{Now } \angle QRO = \angle qRO;$$

$$\therefore \frac{QR}{Rq} = \frac{QO}{Oq}, \quad (\text{Euc. VI. 3.})$$

or ultimately, when R moves up to A and q to F ,

$$AQ \cdot OF = AF \cdot OQ,$$

$$u(r - v) = v(u - r),$$

$$\frac{1}{v} - \frac{1}{r} = \frac{1}{r} - \frac{1}{u},$$

$$\frac{1}{v} + \frac{1}{u} = \frac{2}{r},$$

which determines the position of F .

COR. 1. If u be infinitely great, or the incident pencil consist of parallel rays

$$v = \frac{r}{2},$$

which assigns the position of the principal focus of the reflector (16).

$$\text{COR. 2. Since } \frac{1}{v} + \frac{1}{u} = \frac{2}{r},$$

$$\left(\frac{r}{2}\right)^2 = uv - (u + v)\frac{r}{2} + \left(\frac{r}{2}\right)^2 = \left(u - \frac{r}{2}\right)\left(v - \frac{r}{2}\right);$$

\therefore if F_1 be the principal focus

$$AF_1^2 = F_1Q \cdot F_1F.$$

27. The case of a divergent pencil and concave mirror, by which the proposition has been investigated, is chosen because in it all the lines are measured in a positive direction. It will be seen that attention to the signs of u and r will make this case include every other. Light is always supposed to proceed from right to left, and positive lines consequently to be measured from left to right; hence

- (1) if the surface be convex, or O to the left of A , r is negative,
 (2) pencil...convergent,... Q A , u is negative.

The sign which v has in the result, determines on which side of the point A the geometrical focus of the pencil lies, as the magnitude of v gives its distance from A .

The four cases of the proposition are here subjoined.

$$\text{I. Divergent pencil } \left. \begin{array}{l} \text{Concave mirror} \end{array} \right\} \therefore \left\{ \begin{array}{l} u \text{ positive} \\ r \text{ positive} \end{array} \right. \text{ (fig. 11)}$$

F lies to the right or left of A as u is $\begin{array}{l} > \frac{r}{2} \\ < \frac{r}{2} \end{array}$.

$$\text{II. Divergent pencil } \left. \begin{array}{l} \text{Convex mirror} \end{array} \right\} \therefore \left\{ \begin{array}{l} u \text{ positive} \\ r \text{ negative} \end{array} \right. \text{ (fig. 12)}$$

F lies always to the left of A .

$$\text{III. Convergent pencil } \left. \begin{array}{l} \text{Concave mirror} \end{array} \right\} \therefore \left\{ \begin{array}{l} u \text{ negative} \\ r \text{ positive} \end{array} \right. \text{ (fig. 13)}$$

F always lies to the right of A .

$$\text{IV. Convergent pencil } \left. \begin{array}{l} \text{Convex mirror} \end{array} \right\} \therefore \left\{ \begin{array}{l} u \text{ negative} \\ r \text{ negative} \end{array} \right. \text{ (fig. 14)}$$

$\therefore F$ lies to the right or left of A as u is $\begin{array}{l} < \frac{r}{2} \\ > \frac{r}{2} \end{array}$.

28. Since $\frac{1}{v} + \frac{1}{u} = \frac{2}{r}$, if F were the origin of a pencil, Q would be its geometrical focus after reflection at the spherical mirror. The points Q and F are thus convertible, and for this reason are sometimes called conjugate foci. It also

appears from the same equation that these foci move in contrary directions, and that if one of them coincide with the principal focus of the reflector the other is indefinitely distant.

29. When a pencil is incident directly on a spherical reflecting surface to find the point where the direction of a given ray after reflection cuts the axis.

Let Q (fig. 16) be the origin of a pencil whose axis QA is incident directly at A on a spherical reflecting surface whose center is O ; then AQ is the axis of the reflected pencil. Let QR be any ray incident at R and reflected in a direction which cuts the axis in q , the position of which is to be determined. Join OR .

Let $AQ = u$, $Aq = v'$, $AO = r$, lines being considered positive when measured from A in a direction opposite to that of the incident pencil. Also let $AR = y$, a quantity of which the cube and higher powers may be neglected.

Then since $\angle QRq$ is bisected by RO ,

$$\frac{QR}{qR} = \frac{QO}{qO}, \text{ or } QR \cdot qO = qR \cdot QO.$$

$$\text{Now } QR^2 = RO^2 + OQ^2 + 2OR \cdot OQ \cos ROA,$$

$$= r^2 + (u - r)^2 + 2r(u - r) \cos \frac{y}{r},$$

$$= u^2 - \frac{u - r}{r} \cdot y^2; \quad \therefore \cos \frac{y}{r} = 1 - \frac{y^2}{2r^2};$$

$$\therefore QR = u \left\{ 1 - \left(\frac{1}{r} - \frac{1}{u} \right) \frac{y^2}{2u} \right\}.$$

$$\text{So } qR = v' \left\{ 1 - \left(\frac{1}{r} - \frac{1}{v'} \right) \frac{y^2}{2v'} \right\}.$$

$$\therefore (r - v') u \left\{ 1 - \left(\frac{1}{r} - \frac{1}{u} \right) \frac{y^2}{2u} \right\} = (u - r) v' \left\{ 1 - \left(\frac{1}{r} - \frac{1}{v'} \right) \frac{y^2}{2v'} \right\}.$$

If each side of this equation be divided by urv' ,

$$\left(\frac{1}{v'} - \frac{1}{r}\right) \left\{1 - \left(\frac{1}{r} - \frac{1}{u}\right) \frac{y^2}{2u}\right\} = \left(\frac{1}{r} - \frac{1}{u}\right) \left\{1 - \left(\frac{1}{r} - \frac{1}{v'}\right) \frac{y^2}{2v'}\right\},$$

$$\text{or } \frac{1}{v'} + \frac{1}{u} = \frac{2}{r} + \left(\frac{1}{r} - \frac{1}{u}\right) \left(\frac{1}{v'} - \frac{1}{r}\right) \left(\frac{1}{u} + \frac{1}{v'}\right) \frac{y^2}{2}.$$

If we neglect higher powers of y than the first,

$$\frac{1}{v'} + \frac{1}{u} = \frac{2}{r},$$

$$\text{or } v' = v,$$

which value we may substitute in the coefficient of y^2 and the resulting equation will be true as far as y^2 .

$$\therefore \frac{1}{v'} + \frac{1}{u} = \frac{2}{r} + \left(\frac{1}{r} - \frac{1}{u}\right)^2 \frac{y^2}{r},$$

which gives the position of q .

$$30. \quad \text{COR.} \quad \frac{1}{v'} - \frac{1}{v} = \left(\frac{1}{r} - \frac{1}{u}\right)^2 \frac{y^2}{r};$$

$$\therefore v' - v = - \left(\frac{1}{r} - \frac{1}{u}\right)^2 \frac{y^2}{r} \cdot vv',$$

or putting $v' = v$ in a term involving y^2 we have

$$\text{aberration of ray } Rq = v' - v = - \left(\frac{1}{r} - \frac{1}{u}\right)^2 \frac{v^2 y^2}{r}$$

$$= - \frac{\left(\frac{1}{r} - \frac{1}{u}\right)^2 y^2}{\left(\frac{2}{r} - \frac{1}{u}\right)^2 r}.$$

31. It is sometimes convenient to determine the position of the geometrical focus of a reflected pencil by its distance from the center of the reflecting surface instead of the point of incidence.

Let Q (fig. 15) be the origin of a pencil of light whose axis QA is incident directly at A , on a spherical reflecting surface, whose center is O . Then AQ is the axis of the reflected pencil. Let QR be any ray incident at R and reflected in a direction which cuts OQ in q , F the geometrical focus of the reflected pencil.

Let $OQ = p$, $OA = r$, $OF = q$, lines being considered positive when measured from O in a direction opposite to that of the incident pencil. Also let $ROA = \theta$, ϕ = angle of incidence or reflection of QR .

$$\text{Then } \frac{r}{Oq} = \frac{\sin RqA}{\sin ORq} = \frac{\sin (\phi + \theta)}{\sin \phi},$$

$$\frac{r}{p} = \frac{\sin RQA}{\sin ORQ} = \frac{\sin (\phi - \theta)}{\sin \phi};$$

$$\therefore \frac{r}{Oq} + \frac{r}{p} = \frac{\sin (\phi + \theta) + \sin (\phi - \theta)}{\sin \phi} = 2 \cos \theta.$$

In the limit $Oq = q$, and $\cos \theta = 1$,

$$\therefore \frac{r}{q} + \frac{r}{p} = 2,$$

$$\frac{1}{q} + \frac{1}{p} = \frac{2}{r},$$

which gives the position of F .

OBS. It will be seen that in investigating the above equation the case of a convex mirror has been taken in order that all the lines which enter may be measured in a positive direction.

COR. If as before $AR = y$, whose cube and higher powers may be neglected,

$$\frac{r}{q} - \frac{r}{Oq} = 2 \left(1 - \cos \frac{y}{r} \right) = \frac{y^2}{r^2};$$

$$\therefore Oq - q = \frac{q^2 y^2}{r^3},$$

which is the aberration of the ray qR , and agrees with the expression in (30).

32. To find the geometrical focus of a pencil of rays after direct refraction at a spherical surface.

Let Q (fig. 17) be the origin of a pencil of light whose axis QA is incident directly on a spherical refracting surface of which O is the center; then QA is the axis of the refracted pencil. Let QR be any ray incident at R and refracted in a direction which cuts QA in q , F the geometrical focus. Join OR .

Let $AQ = u$, $AO = r$, $AF = v$, lines being considered positive when measured from A in a direction contrary to that of the incident pencil.

$$\text{Now } \mu = \frac{\sin QRO}{\sin qRO} = \frac{\sin QRO}{\sin ROQ} \cdot \frac{\sin ROq}{\sin qRO} = \frac{QO}{RQ} \cdot \frac{Rq}{Oq}.$$

$$\text{In the limit } \mu = \frac{QO}{AQ} \cdot \frac{AF}{OF},$$

$$\mu u (v - r) = (u - r) v,$$

$$\mu \left(\frac{1}{r} - \frac{1}{v} \right) = \frac{1}{r} - \frac{1}{u},$$

$$\frac{\mu}{r} - \frac{1}{u} = \frac{\mu - 1}{r},$$

which determines the position of F .

COR. The position of the principal focus of the surface is given by the formula

$$v = \frac{\mu}{\mu - 1} r.$$

33. To find the distance of the geometrical focus of a pencil of rays from the center after direct refraction at a spherical surface.

Let Q (fig. 18) be the origin of a pencil of light whose axis QA is incident directly at A on a spherical refracting surface whose center is O ; then QA is the axis of the refracted pencil. Let QR be any ray incident at R and re-

fracted in a direction which cuts AQ in q , F the geometrical focus of the refracted pencil.

Let $OQ = p$, $AO = r$, $OF = q$, lines being considered positive when measured from O in a direction contrary to that of the incident pencil.

$$\text{Then } \mu = \frac{\sin QRO}{\sin qRO} = \frac{\sin QRO}{\sin ROQ} \cdot \frac{\sin ROq}{\sin qRO} = \frac{OQ}{RQ} \cdot \frac{Rq}{Oq}.$$

$$\text{In the limit } \mu = \frac{OQ}{AQ} \cdot \frac{AF}{OF},$$

$$\text{or } \mu \cdot AQ \cdot OF = OQ \cdot AF;$$

$$\therefore \mu q (p - r) = p (q - r),$$

$$\mu \left(\frac{1}{r} - \frac{1}{p} \right) = \frac{1}{r} - \frac{1}{q},$$

$$\frac{1}{q} - \frac{\mu}{p} = - \frac{\mu - 1}{r},$$

which determines the distance of F from O .

34. When a pencil is incident directly on a spherical refracting surface to find the point where the direction of a given ray after refraction cuts the axis.

Let Q (fig. 19) be the origin of a pencil whose axis QA is incident directly at A on a spherical refracting surface whose center is O ; then QA is the axis of the refracted pencil. Let QR be any ray incident at R , and refracted in a direction which cuts the axis in q , the position of which is to be determined. Join OR .

Let $AQ = u$, $Aq = v'$, $AO = r$, lines being considered positive when measured from A in a direction opposite to that of the incident pencil. Also let $AR = y$, a quantity of which the cube and higher powers may be neglected.

$$\text{Then } \mu = \frac{\sin QRO}{\sin ROQ} \cdot \frac{\sin ROq}{\sin qRO} = \frac{OQ}{RQ} \cdot \frac{Rq}{Oq},$$

$$\text{or } \mu \cdot RQ \cdot Oq = Rq \cdot OQ.$$

Now $RQ^2 = RO^2 + OQ^2 + 2RO \cdot OQ \cos ROA,$

$$= r^2 + (u - r)^2 + 2r(u - r) \cos \frac{y}{r},$$

$$= u^2 - (u - r) \frac{y^2}{r};$$

$$\therefore RQ = u \left\{ 1 - \left(\frac{1}{r} - \frac{1}{u} \right) \frac{y^2}{2u} \right\}.$$

$$\text{So } Rq = v' \left\{ 1 - \left(\frac{1}{r} - \frac{1}{v'} \right) \frac{y^2}{2v'} \right\}.$$

$$\therefore \mu (v' - r) u \left\{ 1 - \left(\frac{1}{r} - \frac{1}{u} \right) \frac{y^2}{2u} \right\} = (u - r) v' \left\{ 1 - \left(\frac{1}{r} - \frac{1}{v'} \right) \frac{y^2}{2v'} \right\},$$

$$\mu \left(\frac{1}{r} - \frac{1}{v'} \right) \left\{ 1 - \left(\frac{1}{r} - \frac{1}{u} \right) \frac{y^2}{2u} \right\} = \left(\frac{1}{r} - \frac{1}{u} \right) \left\{ 1 - \left(\frac{1}{r} - \frac{1}{v'} \right) \frac{y^2}{2v'} \right\},$$

$$\frac{\mu}{v'} - \frac{1}{u} = \frac{\mu - 1}{r} + \left(\frac{1}{r} - \frac{1}{v'} \right) \left(\frac{1}{r} - \frac{1}{u} \right) \left(\frac{1}{v'} - \frac{\mu}{u} \right) \frac{y^2}{2}.$$

If the square of y be neglected, $v' = v$ is given by the equation

$$\frac{\mu}{v} - \frac{1}{u} = \frac{\mu - 1}{r},$$

which value of v' may be used in the coefficient of y^2 and the result will be true as far as y^2 .

$$\therefore \frac{\mu}{v'} - \frac{1}{u} = \frac{\mu - 1}{r} + \frac{1}{\mu^2} \left(\frac{1}{r} - \frac{1}{u} \right)^2 \left(\frac{1}{u} + \frac{\mu - 1}{r} - \frac{\mu^2}{u} \right) \frac{y^2}{2},$$

$$= \frac{\mu - 1}{r} + \frac{\mu - 1}{\mu^2} \left(\frac{1}{r} - \frac{1}{u} \right)^2 \left(\frac{1}{r} - \frac{\mu + 1}{u} \right) \frac{y^2}{2},$$

which determines the position of q .

$$35. \quad \text{COR.} \quad \frac{\mu}{v'} - \frac{\mu}{v} = \frac{\mu - 1}{\mu^2} \left(\frac{1}{r} - \frac{1}{u} \right)^2 \left(\frac{1}{r} - \frac{\mu + 1}{u} \right) \frac{y^2}{2}.$$

Therefore the aberration of the ray $Rq = v' - v$

$$\begin{aligned} &= -\frac{\mu - 1}{\mu^3} \left(\frac{1}{r} - \frac{1}{u} \right)^2 \left(\frac{1}{r} - \frac{\mu + 1}{u} \right) \frac{y^2 v^2}{2} \\ &= -\frac{\mu - 1}{\mu} \frac{\left(\frac{1}{r} - \frac{1}{u} \right)^2}{\left(\frac{\mu - 1}{r} + \frac{1}{u} \right)^2} \left(\frac{1}{r} - \frac{\mu + 1}{u} \right) \frac{y^2}{2}. \end{aligned}$$

Least Circle of Aberration.

36. Let AF (fig. 20) be the axis of a pencil of rays reflected or refracted directly at a spherical surface, Ar , hr the extreme rays in any plane section of the pencil through the axis which meet in the point r of the axis, Ks , ks any other two rays in the same section meeting in the point s of the axis. Suppose hr produced to cut Ks in t and draw tm perpendicular to AF .

Now (1) since when K coincides with either A or H $tm = 0$, therefore for some position of K in AH tm is a maximum.

(2) If K have the position for which tm is a maximum, a circle with center m and radius mt in a plane perpendicular to AF is the smallest space through which the whole pencil passes. For a circular section of the pencil to the left of m is larger in consequence of the converging cone Ksk and a circular section to the right is larger in consequence of the diverging cone trm . Such a circle is called the least Circle of Aberration of the reflected or refracted pencil.

In the following calculation of the position and dimensions of this circle the cube and higher powers of the half breadth of the pencil will be neglected, as has been done in former cases.

37. To calculate the position and dimensions of the least circle of aberration after direct reflection or refraction, at a plane or spherical surface.

In the figure of Art. (36) let $AK = y'$, $AH = y$, r = the radius of the spherical surface AK . Draw KM perpendicular to AF .

$$\therefore KM = r \sin \frac{y'}{r} = y',$$

$$AM = r \text{ vers } \frac{y'}{r} = \frac{y'^2}{2r}.$$

By similar triangles

$$\frac{ms}{tm} = \frac{Ms}{KM} = \frac{As - \frac{y'^2}{2r}}{y'} = \frac{As}{y'} - \frac{y'}{2r};$$

$$\therefore ms = tm \cdot \frac{Ar}{y'};$$

for tm depending on the aberration is of the order of y'^2 ; $\therefore y' \cdot tm$ may be neglected; also the difference of Ar and As will be when multiplied by tm of a higher order than y'^3 .

$$\text{So } \frac{mr}{tm} = \frac{Mr}{Ah} = \frac{Ar}{y} - \frac{y}{2r};$$

$$mr = tm \cdot \frac{Ar}{y};$$

$$\therefore rs = rm + ms = tm \cdot Ar \left\{ \frac{1}{y} + \frac{1}{y'} \right\}.$$

But $Fs : Fr :: y'^2 : y^2$ (25. 30. 35).

$$\therefore rs : Fr :: y^2 - y'^2 : y^2;$$

$$\therefore Fr \frac{y^2 - y'^2}{y^2} = tm \cdot Ar \cdot \frac{y + y'}{yy'}$$

$$tm \cdot Ar = Fr \cdot \frac{(y - y') y'}{y}.$$

If y' be such that tm is a maximum,

$$y - 2y' = 0, \quad y' = \frac{y}{2};$$

$$\therefore tm \cdot Ar = Fr \cdot \frac{y}{4}.$$

$$tm = \frac{1}{4} \frac{Fr}{Ar} \cdot y \dots \dots \dots (1),$$

$$rm = \frac{1}{4} \cdot Fr \dots \dots \dots (2),$$

which define the position and magnitude of the least circle of aberration.

The distance of the circle from the geometrical focus is thus three fourths of the aberration of the pencil and its radius is one fourth of the distance of the extreme rays of the pencil from the geometrical focus measured perpendicularly to the axis.

38. By the intersection of consecutive reflected or refracted rays in the plane AHF (fig. 20) a curve is generated. The surface produced by the revolution of this curve about AF is the caustic surface of the reflected or refracted pencil. The point F is the vertex of the surface.

The determination of the caustic surface of a pencil for a given reflecting or refracting surface of revolution is a geometrical problem of which the following is the general process. In any section through the axis of the pencil the equation to any reflected or refracted ray referred to axes in that plane will involve two parameters, the co-ordinates of the point of incidence, connected by the equation to the surface. The curve formed by the intersections of such consecutive rays may therefore be found (Miller's Diff. Calc. 119.) and the caustic surface is generated by the revolution of this curve.

39. The reflecting or refracting surfaces have been considered spherical because such are the surfaces which generally occur in the construction of instruments. When the surface is any other figure of revolution, it is in general sufficient when the pencil is not very large to consider the surface the same as the spherical surface generated by the circle of curvature at the vertex of the generating curve.

Such cases are in general questions of curiosity. When however the reflecting or refracting surface is generated by the revolution of a conic section, there are cases of the reflection or refraction of pencils without aberration which it is fit to notice since these properties have been employed in practice.

40. A pencil of parallel rays incident parallel to its axis on a reflecting paraboloid of revolution will be reflected without aberration.

Let A, S (fig. 21) be the vertex and focus of a paraboloid of revolution, QR a line parallel to the axis, RG the normal at the point R . Join SR . Then QR and SR or SR produced lie in the same plane with the normal at R and make equal angles with it. If therefore QR be a ray of light incident parallel to the axis on the reflecting paraboloid its direction after reflection passes through S . Thus a pencil of rays parallel to the axis of the paraboloid after reflection converges to or diverges from one point S .

41. A pencil of rays diverging from or converging to a focus of a spheroid or hyperboloid of revolution will be reflected without aberration.

Let H, S (fig. 22) be the foci of a reflecting spheroid or hyperboloid of revolution of which R is any point. Join SR, HR . Then SR and HR or those lines produced lie in the same plane with the normal to the surface at R and make equal angles on opposite sides of it. If therefore one of these lines be the direction of a ray of light incident on the surface the other line will be the direction of the reflected ray. Thus a pencil of rays converging to or diverging from one focus of the surface will after reflection diverge from or converge to the other focus.

42. A pencil of parallel rays, incident parallel to the axis on a refracting spheroid or hyperboloid of revolution may be refracted without aberration.

Let QR (fig. 23) be a line parallel to the axis of a spheroid or hyperboloid of revolution, S the farther focus,

RG the normal at R . Join SR which lies in the same plane with QR , RG . If e be the eccentricity,

$$\frac{\sin QRG}{\sin SRG} = \frac{SR}{SG} = \frac{1}{e}.$$

Therefore if $\mu = \frac{1}{e}$, a ray QR incident on the spheroid or hyperboloid will be refracted in a direction which passes through S . Thus a pencil of rays parallel to the axis converges to or diverges from one point S after refraction.

SECTION II.

OBLIQUE REFLECTION AND REFRACTION OF LIGHT.

43. OBLIQUE incidence on a reflecting or refracting spherical surface is either centrical or excentrical. A pencil is incident centrically when the axis of the pencil is incident at a definite point of the surface called the center of its face; in other cases it is incident excentrically.

A distinction must be preserved between the center of the surface and the center of the face; the former is that point from which every point of the surface is equidistant, the latter is a point of the surface.

DEF. The diameter through the center of the face is in general called the axis of the reflecting or refracting surface.

The circumstance which renders the calculations of centrical pencils less complex than those of excentrical pencils is that in the former case the point of incidence can be used as a point of reference from which lines may be measured, but in the latter it cannot.

44. If a divergent pencil be incident obliquely on a plane reflecting surface it will diverge from a point after reflection.

Let Q (fig. 24) be the origin of a pencil whose axis QC is incident directly on a plane reflecting surface CH ; then after reflection the pencil will diverge from a point q in QC produced at a distance $Cq = CQ$. (20). If now the whole pencil be supposed removed excepting the oblique

pencil QHK whose axis is QA the course of this portion of the pencil will remain unaltered, or the oblique pencil will after reflection diverge from q .

COR. Since $AqC = AQC$, and $KqC = KQC$ (20);

$$\therefore AqK = AqK;$$

or the degree of divergence of the reflected pencil is the same as that of the incident pencil. (5).

45. In other cases of oblique reflection or refraction the reflected or refracted pencil if it be small will be shewn to converge to or diverge from two very small straight lines perpendicular to one another and in different planes so that the direction of every ray passes through each of these straight lines.

46. To explain the formation of focal lines when a small oblique pencil is reflected at a spherical surface, or refracted at a plane or spherical surface.

Let QC (fig. 25) be the axis of a pencil incident directly on a spherical reflecting surface, or on a plane or spherical refracting surface. If this pencil be supposed to consist of a series of conical surfaces of rays with QC for their common axis, since all the rays in any such surface will be reflected or refracted similarly about QC , the directions of the reflected or refracted rays will form a series of conical surfaces Hsh , Aq_2a , Krk having a common axis QC along which their vertices r , q_2 , s ,...are arranged. The intersection of these consecutive conical surfaces produces the caustic surface. (38).

(1) If we consider instead of the whole pencil that portion of it only which is incident on the annulus of the reflecting or refracting surface which would be generated by the revolution of HK about QC , we have corresponding to this an annulus of the caustic surface through some point of which the direction of each ray of the conical shell of reflected or refracted rays now considered passes. If HK

be small, this annulus of the caustic surface may be considered a circle q_1t in a plane perpendicular to QC .

(2) In place of the conical shell of light incident on the above annulus let us consider a small portion only thereof incident about HK , by which we come to the case of a small oblique pencil whose axis after reflection is Aq_1q_2 . The direction of every reflected or refracted ray will now pass through some point in a small circular arc at q_1 which may approximately be considered a straight line perpendicular to the plane QCA . This line is called the Primary Focal line and the point q_1 the Primary Focus. Again a section of the pencil by a plane through q_2 parallel to the tangent plane of the surface at A , though actually a very elongated figure of eight (fig. 26) may very approximately be regarded as a straight line and is called the Secondary Focal line, while the point q_2 where the axis of the reflected or refracted pencil cuts QC is called the Secondary Focus.

DEF. The plane QCA or plane of incidence of the axis of the pencil is called the Primary Plane.

Hence a small oblique pencil after reflection or refraction converges to or diverges from two straight lines one in and the other perpendicular to the primary plane.

When the aberration of a direct pencil is towards the surface, the primary focus of a small oblique pencil from the same origin is nearer to the surface than the secondary focus, and vice versâ.

47. Circle of Least Confusion.

If a section be taken of the reflected or refracted pencil by a plane parallel to the tangent plane to the reflecting or refracting surface at A , when the plane is drawn through q_1 the section as we have seen is a straight line perpendicular to the primary plane. If the point through which the plane is drawn be supposed gradually to move from q_1 to q_2 , the breadth of the section increases in the primary plane and decreases in the perpendicular direction until at q_2 the section becomes a straight line in

the primary plane. At some point therefore in q_1q_2 the two dimensions of the section are equal and the section very nearly circular. This section of the pencil is called the Circle of Least Confusion.

48. If a small oblique pencil be reflected at a spherical surface, to find the distances of the foci from the point of incidence of the axis.

Let Q (fig. 27) be the origin of a small pencil whose axis QA is incident at A obliquely on a spherical reflecting surface whose center is O . Let Aq_2 be the direction of the axis after reflection cutting QO produced in q_2 the secondary focus (46), QH another ray incident in the primary plane and reflected in direction Hq_1 which cuts Aq_2 in q_1 the primary focus. Join OA , OH , and draw Hn perpendicular to AQ . Let QA , HO intersect in K .

Let $AQ = u$, $Aq_1 = v_1$, $Aq_2 = v_2$, $AO = r$, $AOq_2 = \theta$,

$\left. \begin{matrix} \phi \\ \phi + \delta\phi \end{matrix} \right\}$ the angles of incidence or reflection of $\left\{ \begin{matrix} QA \\ QH \end{matrix} \right.$

$$\begin{aligned} \text{Then } \angle HQA &= \frac{Hn}{HQ} \text{ ultimately,} \\ &= \frac{AH \sin HAQ}{HQ} \text{ ultimately,} \\ &= \frac{AH \cos \phi}{u} \text{ ultimately.} \end{aligned}$$

$$\begin{aligned} \therefore \delta\phi &= QHO - QAO, \\ &= K - AQH - (K - AOH), \\ &= AOH - AQH, \\ &= \frac{AH}{r} - \frac{AH \cos \phi}{u}. \end{aligned}$$

$$\begin{aligned} \text{So } \delta\phi &= q_1HO - q_1AO, \\ &= \frac{AH \cos \phi}{v_1} - \frac{AH}{r}. \end{aligned}$$

$$\therefore \frac{\cos \phi}{v_1} - \frac{1}{r} = \frac{1}{r} - \frac{\cos \phi}{u},$$

$$\frac{1}{v_1} + \frac{1}{u} = \frac{2}{r \cos \phi} \quad (1).$$

$$\text{Again } \frac{r}{v_2} = \frac{AO}{Aq_2} = \frac{\sin(\theta + \phi)}{\sin \theta},$$

$$\frac{r}{u} = \frac{AO}{AQ} = \frac{\sin(\theta - \phi)}{\sin \theta};$$

$$\therefore \frac{r}{v_2} + \frac{r}{u} = \frac{\sin(\theta + \phi) + \sin(\theta - \phi)}{\sin \theta},$$

$$= 2 \cos \phi;$$

$$\frac{1}{v_2} + \frac{1}{u} = \frac{2 \cos \phi}{r} \quad (2).$$

Equations (1) and (2) give respectively the distances of the primary and secondary foci from A .

49. If the point A be the center of the face of the reflecting surface, the positions of the foci of the reflected pencil are conveniently determined by the distance from A of their projections on the axis of the mirror AO .

Draw QM , q_1m_1 , q_2m_2 perpendicular to AO .

Let $AM = h$, $Am_1 = k_1$, $Am_2 = k_2$.

$$\therefore u = h \sec \phi, \quad v_1 = k_1 \sec \phi, \quad v_2 = k_2 \sec \phi.$$

Hence equations (1) and (2) of (48) give

$$\frac{\cos \phi}{k_1} + \frac{\cos \phi}{h} = \frac{2}{r \cos \phi}$$

$$\left. \begin{aligned} \frac{1}{k_1} + \frac{1}{h} &= \frac{2}{r \cos^2 \phi} \\ \frac{1}{k_2} + \frac{1}{h} &= \frac{2}{r} \end{aligned} \right\}.$$

If ϕ be so small that powers of it above the square may be neglected,

$$\frac{1}{k_1} + \frac{1}{h} = \frac{2}{r} (1 + \phi^2).$$

$$\text{Let } \varepsilon = q_2 m_2, \quad \therefore \frac{\varepsilon}{k_2} = \tan \phi = \phi,$$

an approximation which being employed in the above equation gives

$$\frac{1}{k_2} + \frac{1}{h} = \frac{2}{r} + \frac{2\varepsilon^2}{rk_2^2},$$

where $\frac{1}{k_2}$ is known from the equation

$$\frac{1}{k_2} + \frac{1}{h} = \frac{2}{r}.$$

50. When a small pencil is incident obliquely on a plane refracting surface to find the distances of the focal lines from the point of incidence of the axis.

Let Q (fig. 28) be the origin of a small pencil whose axis QA is incident obliquely at A on a plane refracting surface. Draw QC perpendicular to the surface, and produce it backwards to cut the direction of QA after refraction in q_2 the secondary focus. Let QH be any ray in the primary plane whose direction after refraction cuts Aq_2 produced in q_1 the primary focus. Draw Hn perpendicular to QA .

$$\text{Let } AQ = u, \quad Aq_1 = v_1, \quad Aq_2 = v_2,$$

$$\left. \begin{array}{l} \phi \\ \phi + \delta\phi \end{array} \right\} \text{ angles of incidence of } \left\{ \begin{array}{l} QA \\ QH \end{array} \right.,$$

$$\left. \begin{array}{l} \phi' \\ \phi' + \delta\phi' \end{array} \right\} \dots\dots\dots \text{refraction} \dots\dots\dots$$

$$\text{Now } \delta\phi = HQA = \frac{Hn}{HQ} \text{ approximately, } = \frac{AH \cos \phi}{u}.$$

$$\delta\phi' = Hq_1 A = \frac{AH \cos \phi'}{v_1};$$

$$\therefore \frac{\delta\phi}{\delta\phi'} = \frac{\frac{\cos \phi}{u}}{\frac{\cos \phi'}{v_1}}.$$

$$\text{But } \sin(\phi + \delta\phi) = \mu \sin(\phi' + \delta\phi'), \quad \sin \phi = \mu \sin \phi';$$

$$\therefore \sin(\phi + \delta\phi) - \sin \phi = \mu \{\sin(\phi' + \delta\phi') - \sin \phi'\},$$

$$\cos\left(\phi + \frac{\delta\phi}{2}\right) \sin \frac{\delta\phi}{2} = \mu \cos\left(\phi' + \frac{\delta\phi'}{2}\right) \sin \frac{\delta\phi'}{2}.$$

$$\therefore \frac{\delta\phi}{\delta\phi'} = \frac{\sin \frac{\delta\phi}{2}}{\sin \frac{\delta\phi'}{2}} \text{ ultimately, } = \frac{\mu \cos \phi'}{\cos \phi} *.$$

$$\therefore \frac{\frac{\cos \phi}{u}}{\frac{\cos \phi'}{v_1}} = \frac{\mu \cos \phi'}{\cos \phi},$$

$$\text{or } \frac{\mu \cos^2 \phi'}{v_1} - \frac{\cos^2 \phi}{u} = 0 \quad (1).$$

$$\text{Again } \frac{u}{v_2} = \frac{AQ}{Aq_2} = \frac{\sin \phi'}{\sin \phi} = \frac{1}{\mu};$$

$$\therefore \frac{\mu}{v_2} - \frac{1}{u} = 0 \quad (2).$$

51. When a small pencil is obliquely refracted at a spherical surface to find the distances of the foci from the point of incidence of the axis.

Let Q (fig. 29) be the origin of a small pencil whose axis QA is incident obliquely at A on a spherical refracting

* The reader who is acquainted with Differential Calculus may more readily obtain this value of $\frac{\delta\phi}{\delta\phi'}$ by differentiating the equation $\sin \phi = \mu \sin \phi'$.

surface whose center is O . Let QO cut the direction of QA after refraction in q_2 the secondary focus. Let QH be any ray in the primary plane whose direction after refraction cuts Aq_2 in q_1 the primary focus. Join AO , HO , and draw Hn perpendicular to AQ . Let OH , AQ intersect in K .

Let $AQ = u$, $Aq_1 = v_1$, $Aq_2 = v_2$, $AO = r$, $AOq_2 = \theta$,

$\left. \begin{matrix} \phi \\ \phi + \delta\phi \end{matrix} \right\}$ angles of incidence of $\left\{ \begin{matrix} QA \\ QH \end{matrix} \right.$,

$\left. \begin{matrix} \phi' \\ \phi' + \delta\phi' \end{matrix} \right\}$ refraction

$$\therefore HQA = \frac{Hn}{HQ} \text{ ultimately, } = \frac{AH \cos \phi}{u} \text{ ultimately.}$$

$$\begin{aligned} \text{Now } \delta\phi &= QHO - QAO, \\ &= K - HQA - (K - HOA), \\ &= HOA - HQA, \\ &= \frac{AH}{r} - \frac{AH \cos \phi}{u}. \end{aligned}$$

$$\text{So } \delta\phi' = \frac{AH}{r} - \frac{AH \cos \phi'}{v_1}.$$

$$\therefore \frac{\delta\phi}{\delta\phi'} = \frac{\frac{\cos \phi}{u} - \frac{1}{r}}{\frac{\cos \phi'}{v_1} - \frac{1}{r}}.$$

$$\text{But } \sin(\phi + \delta\phi) = \mu \sin(\phi' + \delta\phi'), \quad \sin \phi = \mu \sin \phi';$$

$$\sin(\phi + \delta\phi) - \sin \phi = \mu \{ \sin(\phi' + \delta\phi') - \sin \phi' \},$$

$$\cos\left(\phi + \frac{\delta\phi}{2}\right) \sin \frac{\delta\phi}{2} = \mu \cos\left(\phi' + \frac{\delta\phi'}{2}\right) \sin \frac{\delta\phi'}{2},$$

$$\therefore \frac{\delta \phi}{\delta \phi'} = \frac{\sin \frac{\delta \phi}{2}}{\sin \frac{\delta \phi'}{2}}, \text{ ultimately, } = \frac{\mu \cos \phi'}{\cos \phi}.$$

$$\therefore \frac{\mu \cos \phi'}{\cos \phi} = \frac{\frac{\cos \phi}{u} - \frac{1}{r}}{\frac{\cos \phi'}{v_1} - \frac{1}{r}};$$

$$\therefore \frac{\mu \cos^2 \phi'}{v_1} - \frac{\cos^2 \phi}{u} = \frac{\mu \cos \phi' - \cos \phi}{r} \quad (1).$$

$$\text{Again } \frac{r}{v_2} = \frac{AO}{AQ_2} = \frac{\sin(\phi' + \theta)}{\sin \theta} = \cos \phi' + \sin \phi' \cot \theta,$$

$$\frac{r}{u} = \frac{AO}{AQ} = \frac{\sin(\phi + \theta)}{\sin \theta} = \cos \phi + \sin \phi \cot \theta;$$

$$\therefore \frac{\mu r}{v_2} - \frac{r}{u} = \mu \cos \phi' - \cos \phi,$$

$$\frac{\mu}{v_2} - \frac{1}{u} = \frac{\mu \cos \phi' - \cos \phi}{r} \quad (2).$$

52. If A be the center of the face of the refracting surface and AO its axis, the distances from A of the projections of the foci on the axis may thus be found.

Draw QM , $q_1 m_1$, $q_2 m_2$, perpendicular to AO ,

and let $AM = h$, $Am_1 = k_1$, $Am_2 = k_2$,

Then $u = h \sec \phi$, $v_1 = k_1 \sec \phi'$, $v_2 = k_2 \sec \phi'$.

Equations (1) and (2) of (51) give

$$\frac{\mu \cos^3 \phi'}{k_1} - \frac{\cos^3 \phi}{h} = \frac{\mu \cos \phi' - \cos \phi}{r},$$

$$\frac{\mu \cos \phi'}{k_2} - \frac{\cos \phi}{h} = \frac{\mu \cos \phi' - \cos \phi}{r}.$$

$$\therefore \frac{1}{k_1} = \frac{1}{\mu \cos^3 \phi'} \left\{ \frac{\mu \cos \phi' - \cos \phi}{r} + \frac{\cos^3 \phi}{h} \right\} \quad (1)$$

$$\frac{1}{k_2} = \frac{1}{\mu \cos \phi'} \left\{ \frac{\mu \cos \phi' - \cos \phi}{r} + \frac{\cos \phi}{h} \right\} \quad (2).$$

53. To calculate the position and dimensions of the circle of least confusion of a small oblique pencil reflected at a spherical surface, or refracted at a plane or spherical surface.

Let A (fig. 30) be the point of incidence of the axis of a small oblique pencil on a spherical reflecting surface or a plane or spherical refracting surface. $KMHN$ the section of the incident pencil made by the surface which will approximately be an ellipse with its major and minor axes HK , MN in and perpendicular to the primary plane. Suppose mq_1n , hq_2k the primary and secondary focal lines, and ros , poq the breadths in and perpendicular to the primary plane of a section of the pencil through a point o of its axis by a plane parallel to the tangent plane to the surface at A .

Let $Aq_1 = v_1$, $Aq_2 = v_2$, $MN = \lambda$,

ϕ the angle of incidence of the axis of the pencil.

Now if the incident pencil be small and its origin distant it may be considered approximately cylindrical. HMN then being a section of it by a plane inclined at an angle $\frac{\pi}{2} - \phi$ to its axis,

$$HK = MN \sec \phi = \lambda \sec \phi.$$

$$\text{By similar triangles } \frac{pq}{\lambda} = \frac{oq_2}{v_2}, \quad \frac{rs}{\lambda \sec \phi} = \frac{oq_1}{v_1}.$$

If $pqr s$ be the circle of least confusion, or $pq = rs$,

$$\sec \phi = \frac{v_1}{v_2} \cdot \frac{oq_2}{oq_1} = \frac{v_1}{v_2} \cdot \frac{v_2 - Ao}{Ao - v_1};$$

$$\therefore Ao = \frac{v_1 v_2 (1 + \cos \phi)}{v_1 \cos \phi + v_2} \quad (1)$$

$$pq = \frac{\lambda}{v_2} \cdot (v_2 - Ao), = \lambda \frac{v_2 - v_1}{v_2 + v_1 \cos \phi} \quad (2),$$

which give the distance of the circle of least confusion from A and its diameter.

54. COR. 1. If the pencil be cylindrical at incidence and be reflected at a spherical surface,

$$\frac{1}{v_1} = \frac{2}{r \cos \phi},$$

$$\frac{1}{v_2} = \frac{2 \cos \phi}{r} \quad (48).$$

$$\therefore Ao = \frac{r}{2} \cdot \frac{(1 + \cos \phi) \cos \phi}{1 + \cos^3 \phi},$$

$$pq = \lambda \frac{\sin^2 \phi}{1 + \cos^3 \phi}.$$

$$55. \text{ COR. 2. } \frac{v_2 - Ao}{Ao - v_1} = \frac{v_2 \sec \phi}{v_1},$$

which approaches to unity as ϕ is diminished.

$$\therefore \text{ultimately } Ao = \frac{1}{2} (v_1 + v_2),$$

or the center of the circle of least confusion has the middle point between the two foci for its limiting position, and it may approximately be supposed to have this position when the obliquity of the pencil is small.

SECTION III.

COMBINED REFLECTIONS AND REFRACTIONS.

56. IN this section it is proposed to examine the modification in direction and form which a pencil undergoes after being reflected or refracted more than once at plane or spherical surfaces.

57. DEF. When a ray has its direction altered by reflection or refraction, its deviation is the angle between its present direction and its original direction produced.

Combined Reflections at plane surfaces.

58. If a pencil be reflected once by each of two plane surfaces to find the deviation of its axis, supposing its course in one plane perpendicular to the intersection of the surfaces.

Let $QRSTV$ (fig. 31) be the course of the axis of a pencil reflected at R and S at two plane surfaces in a plane which cuts these surfaces perpendicularly in CA, CB . Then the angle vTV measured from RTv the direction of QR produced towards the point V is the deviation of the axis of the pencil after the two reflections.

Draw Rm, Sn at right angles to the reflecting planes at R and S .

$$\begin{aligned}\text{Now } vTV &= QRS - RST \quad (\text{Euc. I. 32}), \\ &= 2SRm - 2RSn, \quad (7)\end{aligned}$$

$$\begin{aligned}
&= 2 \left(\frac{\pi}{2} - SRC \right) - 2 \left(\frac{\pi}{2} - RSB \right), \\
&= 2 (RSB - SRC), \\
&= 2 ACB;
\end{aligned}$$

or the deviation of the axis of the pencil is double the inclination of the reflecting planes.

COR. The degree of divergence of the pencil is unaltered by the reflections (44. Cor.)

59. To find the deviation of the axis of a pencil reflected at two plane surfaces in any manner.

Let radii of a sphere be drawn parallel to the axis of the pencil at incidence on the first surface, after the first, and after the second reflection, their directions being contrary to that of the pencil in each case, and let them meet the surface of the sphere in P, Q, R respectively (fig. 32). Also let radii parallel to normals to the first and second reflecting surface meet the sphere in A, B . Draw the great circles APQ, BQR and join PR and AB by arcs of great circles.

Let ϕ, ψ be the angles of incidence of the axis of the pencil on the first and second surfaces, D its deviation, i the inclination of the surfaces.

$$\therefore D = PR, \quad i = \pi - AB,$$

$$AP = \phi = \pi - AQ \quad (8) \quad \therefore PQ = \pi - 2\phi,$$

$$BQ = \psi = \pi - BR; \quad \therefore RQ = \pi - 2\psi.$$

$$\text{In the triangle } PQR, \quad \cos PQR = \frac{\cos D - \cos 2\phi \cos 2\psi}{\sin 2\phi \sin 2\psi};$$

$$\text{and in the triangle } QAB, \quad \cos AQB = -\frac{\cos i - \cos \phi \cos \psi}{\sin \phi \sin \psi},$$

$$\text{and } PQR + AQB = \pi;$$

$$\therefore \cos D = \cos 2\phi \cos 2\psi + \frac{\sin 2\phi \sin 2\psi}{\sin \phi \sin \psi} (\cos i - \cos \phi \cos \psi),$$

which determines D .

60. If a pencil of light be reflected in any manner at two plane surfaces, the directions of its axis before the first and after the second reflections are inclined at the same angle to the intersection of the two plane surfaces.

Let radii of a sphere be drawn parallel to the axis of the pencil at incidence on the first surface, after the first, and after the second reflection, their directions being contrary to that of the pencil in each case, and let them meet the surface of the sphere in P, Q, R respectively (fig. 32*). Also let radii parallel to the normals to the first and second surface and to the intersection of the surfaces meet the sphere in A, B, I . Draw the great circles APQ, BQR , and join AB, IP, IR, IA, IB , by arcs of great circles. Then I is the pole of AB , $IA = \frac{\pi}{2} = IB$, $AP + AQ = \pi$, $BQ + BR = \pi$.

$$\begin{aligned}
 \therefore \cos IP &= \sin PA \cdot \cos PAI \\
 &= \sin QA \cdot \sin PAB = \sin QA \\
 &= \sin BQ \cdot \sin QBA \\
 &= \sin BR \cdot \cos RBI \\
 &= \cos IR, \\
 \therefore IP &= IR, \quad = 52
 \end{aligned}$$

or the axis of the pencil before the first and after the second reflection is inclined at the same angle to the intersection of the two reflecting surfaces.

COR. Planes drawn through the line of intersection of the surfaces parallel to the direction of the axis of the pencil before the first and after the second reflection include an angle which is double that between the reflecting surfaces.

For if IQ be joined,

$$\begin{aligned}
 PIQ &= \pi - 2AIP, \quad QIR = \pi - 2BIQ; \\
 \therefore PIR &= 2(BIQ - AIP) \\
 &= 2(\pi - AB).
 \end{aligned}$$

Combined Refractions at plane surfaces.

61. In examining the effect on a pencil of refraction through a medium bounded by two given surfaces these two considerations are employed.

(1) The geometrical focus of a direct pencil being the ultimate point of intersection of any refracted ray with the axis, the pencil after one refraction may ultimately be considered to diverge from or converge to this point as an origin.

(2) From the relation between the angle of incidence and the angles of reflection or refraction given by the laws of reflection or refraction, it is seen that if a ray be reflected or refracted in any manner in passing from one point to another, it might pass by the same course reversed from the latter point to the former. Hence when a pencil is emerging from a refracting medium, we can by supposing the course of each ray reversed reduce this case to the more familiar one of a pencil entering a refracting medium.

62. If ϕ be the angle of incidence of a ray of light and ϕ' its angle of refraction into a denser medium, μ the refractive index between the media,

$$\sin \phi = \mu \sin \phi', \quad \text{or} \quad \sin \phi' = \frac{1}{\mu} \sin \phi.$$

Now μ being > 1 (9) $\sin \phi'$ is < 1 , and this equation gives an angle of refraction for any given angle of incidence. Accordingly refraction into a denser medium is always found possible.

If the refraction be from the denser into the rarer medium and if ϕ' be the angle of incidence, ϕ the angle of refraction,

$$\sin \phi = \mu \sin \phi'.$$

This equation does not give an angle of refraction for any given angle of incidence ϕ' unless $\mu \sin \phi'$ do not exceed 1, or

ϕ' do not exceed $\sin^{-1} \frac{1}{\mu}$. Accordingly if the angle of incidence in the denser medium exceed this limit it is found that refraction does not take place, but that the ray is reflected at the surface separating the media.

DEF. The angle $\sin^{-1} \frac{1}{\mu}$ which the angle of incidence in the denser medium must not exceed in order that refraction into a rarer medium may be possible, is called the critical angle of the media between which the refractive index is μ .

63. DEF. A portion of a refracting medium between two parallel plane surfaces is called a plate.

64. It is a result of experiment that when a ray of light passes through any number of media separated by parallel plane surfaces, if any two of these media be identical, the directions of the ray in them are parallel.

65. Let A, B, C, \dots (fig. 33) be a series of media separated by parallel planes. If a ray of light in A be refracted through these media its direction in L will have undergone the same deviation as if it had been at once refracted from A into L .

For suppose M a medium beyond L separated from it by a plane parallel to the other bounding planes, and let the media M and A be the same. Then if $QR, Q'R'$ be two parallel rays in A their directions in M are parallel when one QR has been refracted through B, C, \dots and the other $Q'R'$ refracted at once into L . But the rays might follow the same courses in a reversed direction, in which case the angles of incidence from M to L being the same, the angles of refraction are the same, or the directions of the two rays in L are parallel, and thus have undergone the same deviation from their direction in A .

66. COR. 1. If $\iota_1, \iota_2, \dots, \iota_n$ be the angles of incidence on B, C, \dots, L and ι_{n+1} the angle of refraction into L , μ_1, μ_2, \dots the indices of refraction between the successive media

$$\frac{\sin \iota_1}{\sin \iota_2} = \mu_1, \quad \frac{\sin \iota_2}{\sin \iota_3} = \mu_2, \quad \dots \quad \frac{\sin \iota_n}{\sin \iota_{n+1}} = \mu_n;$$

$$\begin{aligned} \therefore \mu = \text{index of refraction from } A \text{ to } L &= \frac{\sin \iota_1}{\sin \iota_{n+1}} \\ &= \mu_1 \mu_2 \dots \mu_n. \end{aligned}$$

67. COR. 2. If there be three media A, B, C ,

$$\mu = \text{index of refraction from } A \text{ to } C = \mu_1 \mu_2;$$

$$\therefore \mu_2 = \frac{\mu}{\mu_1}.$$

Hence index of refraction from B into C

$$= \frac{\text{index of refraction from } A \text{ to } C}{\text{index of refraction from } A \text{ to } B}.$$

68. To determine the geometrical focus of a pencil after direct refraction through a plate.

Let Q (fig. 34) be the origin of a pencil whose axis QAB passes directly through a refracting plate. Let F_1 and F be the geometrical foci of the pencil after refraction at the first and second surfaces respectively.

$$\text{Let } AQ = u, \quad BF = v, \quad AB = t.$$

From the first refraction (23)

$$AF_1 = \mu u \quad (1).$$

Now F_1 being regarded as an origin a pencil diverging from F_1 after refraction at the second surface has F for its geometrical focus. Hence if the course of the pencil be supposed reversed, a pencil converging to F would after refraction into the plate have F_1 for its geometrical focus; (61)

$$\therefore BF_1 = \mu \cdot BF,$$

$$\text{or } AF_1 + t = \mu v \quad (2)$$

$$\therefore t = \mu v - \mu u,$$

$$v = u + \frac{t}{\mu},$$

which determines the position of F .

69. We have next to investigate to the same degree of approximation as in former propositions the position of the point of intersection of any ray with the axis of the pencil at emergence from the plate.

Let Q (fig. 35) be an origin from which a pencil proceeds whose axis QAB passes directly through a refracting plate. Let $QRST$ be the course of any ray whose directions after one refraction and at emergence cut the axis in q_1 and q .

$$\text{Let } AQ = u, \quad Bq = v', \quad AB = t, \quad AR = y, \quad BS = y'.$$

From the first refraction

$$Aq_1 = \mu u + \frac{(\mu^2 - 1)}{\mu} \frac{y^2}{2u}.$$

If we suppose the course of the ray $QRST$ reversed we have from the second refraction

$$Bq_1 = \mu v' + \frac{(\mu^2 - 1)}{\mu} \frac{y'^2}{2v'}.$$

$$\therefore t = \mu (v' - u) + \frac{\mu^2 - 1}{2\mu} \left(\frac{y'^2}{v'} - \frac{y^2}{u} \right)$$

$$v' = u + \frac{t}{\mu} - \frac{\mu^2 - 1}{2\mu^2} \left(\frac{y'^2}{v'} - \frac{y^2}{u} \right).$$

Now by similar triangles

$$\frac{y'}{y} = \frac{t + Aq_1}{Aq_1} = 1 + \frac{t}{\mu u}.$$

$$\therefore y' = \left(1 + \frac{t}{\mu u} \right) y,$$

and in the small term we may use for v' its first approximate value $u + \frac{t}{\mu}$;

$$\begin{aligned}\therefore v' &= u + \frac{t}{\mu} - \frac{(\mu^2 - 1)}{\mu^2} \left\{ \frac{\left(1 + \frac{t}{\mu u}\right)^2}{u + \frac{t}{\mu}} - \frac{1}{u} \right\} \frac{y^2}{2} \\ &= u + \frac{t}{\mu} - \left(\frac{\mu^2 - 1}{\mu^2} \right) \left\{ \frac{1}{u} \left(1 + \frac{t}{\mu u}\right) - \frac{1}{u} \right\} \frac{y^2}{2}, \\ &= u + \frac{t}{\mu} - \frac{(\mu^2 - 1)}{\mu^3} \frac{t}{u^2} \frac{y^2}{2}.\end{aligned}$$

COR. Aberration of the ray qS

$$\begin{aligned}&= v' - v \\ &= - \frac{(\mu^2 - 1)}{\mu^3} \frac{t}{u^2} \frac{y^2}{2}.\end{aligned}$$

70. When a small pencil is refracted obliquely through a portion of medium bounded by two given surfaces, there is an apparent difficulty at the second refraction in consequence of neither the incident nor the refracted pencil having then a point of divergence or convergence. In the primary plane however the pencil does on account of its smallness after each refraction converge to or diverge from a point, and we may thus apply with reference to this plane propositions which suppose the whole pencil to emanate from a point. Again the rays incident in a plane perpendicular to the primary plane have after each refraction a point of convergence or divergence, and therefore we may use with respect to this plane the propositions before deduced. The form of the emergent pencil is thus determined by finding the foci of two sections of it, one by the primary plane, the other by a plane perpendicular to that plane. These are separated by an interval which is small in the cases occurring in the construction of Optical Instruments where the pencils are of small extent and obliquity.

71. To determine the foci of a small pencil refracted obliquely through a plate.

Let Q (fig. 36) be the origin of a small pencil whose axis $QAST$ passes obliquely through a plate the surfaces of which it cuts at A and S . Let Q_1, Q_2 be the primary and secondary foci of the pencil after one refraction, q_1, q_2 those of the emergent pencil.

Let $AQ = u$, $Sq_1 = v_1$, $Sq_2 = v_2$, $AB = t$ the thickness of the plate, ϕ, ϕ' the angles of incidence and refraction at A and of emergence and incidence at S . (64).

From the first refraction

$$AQ_1 = \frac{\mu \cos^2 \phi'}{\cos^2 \phi} u \quad (50).$$

Now the pencil diverging in the primary plane from Q_1 diverges after refraction at the second surface of the plate from q_1 in the same plane; hence if we suppose the course of the pencil reversed, a pencil converging to q_1 would after refraction into the plate converge in the primary plane to Q_1 ,

$$\therefore SQ_1 = \frac{\mu \cos^2 \phi'}{\cos^2 \phi} v_1.$$

$$\therefore AS \text{ or } t \sec \phi' = \frac{\mu \cos^2 \phi'}{\cos^2 \phi} (v_1 - u),$$

$$v_1 = u + t \frac{\cos^2 \phi}{\mu \cos^3 \phi'}. \quad (1);$$

Similarly

$$AQ_2 = \mu u,$$

$$SQ_2 = \mu v_2,$$

$$\therefore v_2 = u + \frac{1}{\mu} \frac{t}{\cos \phi'}. \quad (2).$$

Equations (1) and (2) determine respectively the primary and secondary foci of the emergent pencil.

72. DEF. A prism is a portion of a refracting medium bounded by two plane surfaces inclined at a finite angle to one another.

73. DEF. The refracting angle of a prism is the inclination of the two planes which bound it, and the straight line in which these planes intersect is the edge of the prism.

74. When a ray of light is refracted out of one medium into another, as the angle of incidence increases the deviation also increases.

Let ϕ , ϕ' be the angles of incidence and refraction of the ray ;

$$\therefore \sin \phi = \mu \sin \phi',$$

$$\frac{\sin \phi - \sin \phi'}{\sin \phi + \sin \phi'} = \frac{\mu - 1}{\mu + 1},$$

$$\tan \frac{\phi - \phi'}{2} = \frac{\mu - 1}{\mu + 1} \tan \frac{\phi + \phi'}{2}.$$

Now if ϕ and consequently ϕ' be increased,

$\tan \frac{\phi + \phi'}{2}$ is increased, $\frac{\phi + \phi'}{2}$ being less than a right angle ;

$\therefore \tan \frac{\phi - \phi'}{2} \dots\dots\dots, \frac{\mu - 1}{\mu + 1}$ being positive,

$\therefore \phi - \phi' \dots\dots\dots, \frac{\phi - \phi'}{2}$ being less than a right angle,

and $\phi - \phi'$ is the deviation of the ray.

75. The axis of a pencil which passes through a prism denser than the surrounding medium in a plane perpendicular to the edge of the prism is turned from the edge of the prism.

Let $ABCD$ (fig. 37, 38, 39) be a plate of refracting medium, and $QRST$ the axis of a pencil refracted through it,

entering and emerging at R and S respectively. Then the angles of incidence at R and of emergence at S are equal, and the deviations equal and towards opposite parts. Let the face CD of the plate turn about a line through S perpendicular to the plane $QRST$ into the position $C'D'$; the plate then becomes a prism with its edge perpendicular to the plane $QRST$. Let ST' be the direction of QRS at emergence; draw SN , SN' perpendicular to CD , $C'D'$ respectively.

1. Let the angle of incidence at S be on the same side of RS as before and be increased, (fig. 37). Now

deviation at $\left. \begin{matrix} R \\ S \end{matrix} \right\}$ is $\left\{ \begin{matrix} \text{towards} \\ \text{from} \end{matrix} \right.$ the edge of the prism.

But since the angle of emergence at S is greater than in the case of the plate, and since the axis of the pencil might be refracted by the same course reversed (61) the deviation at S is now greater than in the former case (74), i.e. is greater than the deviation at R ; therefore the axis of the pencil is bent from the edge of the prism.

2. Let the angle of incidence at S be on the same side of RS as before and be diminished (fig. 38). Now

deviation at $\left. \begin{matrix} R \\ S \end{matrix} \right\}$ is $\left\{ \begin{matrix} \text{from} \\ \text{towards} \end{matrix} \right.$ the edge of the prism.

But since the angle of emergence at S is less than in the case of the plate, and since the axis of the pencil might be refracted by the same course reversed, the deviation at S is now less than in the former case, i.e. is less than the deviation at R ; therefore the axis of the pencil is bent from the edge of the prism.

3. Let the angle of incidence at S lie on the opposite side of RS to the angle of incidence in the case of the plate (fig. 39). Then the deviations at R and S are each from the edge of the prism; therefore the axis of the pencil is bent from the edge of the prism.

COR. If the surrounding medium be denser than the prism, the axis of the pencil is turned towards the edge of the prism.

76. We proceed to determine the direction and form of a small pencil after refraction through a prism by computing the deviation of its axis and the positions of its focal lines.

77. When a pencil is refracted through a prism to find the deviation of its axis, supposing its course in one plane perpendicular to the edge of the prism.

Let $QRST$ (fig. 40, 41) be the course of the axis of the pencil in a plane which cuts the surfaces of the prism perpendicularly in SA , RA . Let QR produced to some point t cut ST or ST produced backwards in r . Also let the normals to the surfaces at R and S meet in n .

Let ϕ , ϕ' be angles of incidence and refraction at R ,

ψ , ψ' emergence and incidence at S ,

$D = trT$ the deviation of the axis,

$\iota = SAR$ the refracting angle of the prism.

If the intersection n of the normals lie within the prism (fig. 40),

$$D = rSR + rRS = \psi - \psi' + \phi - \phi',$$

and since the four angles of a quadrilateral are = 4 right angles,

$$\therefore \iota = \pi - SnR = nSR + nRS = \phi' + \psi'.$$

If the intersection of the normals lie without the prism (fig. 41),

$$\begin{aligned} D &= SRQ - rSR \\ &= \pi - \phi + \phi' - (\pi - \psi + \psi') \\ &= \psi - \psi' - (\phi - \phi'), \end{aligned}$$

and the inclinations of two surfaces being the same as that of their normals,

$$\iota = RnS = \pi - RSn - SRn = \psi' - \phi';$$

therefore in both cases

$$D = \psi - \psi' \pm (\phi - \phi'), \quad \iota = \psi' \pm \phi'.$$

$$\text{Also } \sin \phi = \mu \sin \phi', \quad \sin \psi = \mu \sin \psi',$$

whence if ϕ' ψ ψ' be eliminated D is known in terms of ϕ , ι , μ .

OBS. If the refracting angle of the prism exceed the critical angle of the medium whereof it is composed, the pencil cannot pass through it, but is internally reflected.

78. COR. 1. If ϕ and ψ be each small,

$$D = (\mu - 1) \psi' \pm (\mu - 1) \phi' \text{ approximately,} \\ = (\mu - 1) \iota.$$

79. COR. 2. If the deviation be a minimum,

$$\text{since } D = \psi' \pm \phi - \iota,$$

$$\iota = \psi' \pm \phi';$$

$$\therefore 0 = d_\phi D = d_\phi \psi \pm 1,$$

$$0 = d_\phi \psi' \pm d_\phi \phi'.$$

$$\text{Also } \sin \phi = \mu \sin \phi'; \quad \therefore \cos \phi = \mu \cos \phi' d_\phi \phi';$$

$$\sin \psi = \mu \sin \psi'; \quad \therefore \cos \psi d_\phi \psi = \mu \cos \psi' d_\phi \psi';$$

$$\therefore d_\phi \phi' = \frac{\cos \phi}{\mu \cos \phi'}, \quad d_\phi \psi' = \mp \frac{\cos \psi}{\mu \cos \psi'};$$

$$\therefore 0 = \frac{\cos \phi}{\mu \cos \phi'} - \frac{\cos \psi}{\mu \cos \psi'},$$

$$\therefore (1 - \sin^2 \psi') (1 - \mu^2 \sin^2 \phi') = (1 - \sin^2 \phi') (1 - \mu^2 \sin^2 \psi'),$$

$$\therefore \sin^2 \phi' = \sin^2 \psi',$$

$$\sin \phi' = \pm \sin \psi'.$$

If the upper sign be used in the original equations or $\phi' + \psi' = \iota$,

$$\therefore \phi' = \psi'.$$

If the lower sign be used or $\psi' - \phi' = \iota$,

$$\therefore \phi' = -\psi'.$$

In the second case the circumstance of one of the angles $\psi' \phi'$ having a contrary sign to that of the other indicates that one of the angles of refraction is measured in a contrary way to that supposed in the proposition, or that the refraction is of the nature of that in the first case.

80. To find the deviation of the axis of a pencil refracted in any manner through a prism.

Let radii of a sphere be drawn parallel to the axis of the pencil at incidence on the first surface, after the first, and after the second refraction, their directions being contrary to that of the pencil in each case, and let them meet the surface of the sphere in P, Q, R respectively (fig. 42). Also let radii parallel to the normals to the first and second surfaces of the prism meet the sphere in A, B . Draw the great circles AQP, BRQ and join PR and AB by arcs of great circles.

Let ϕ, ϕ' be the angles of incidence and refraction of the axis of the pencil at the first surface, ψ, ψ' the angles of emergence and incidence at the second surface, D the deviation of the pencil, ι the angle of the prism;

$$\therefore D = PR, \iota = \pi - AB, AP = \phi, AQ = \phi',$$

$$BQ = \pi - \psi', BR = \pi - \psi.$$

In the triangle PQR

$$\cos PQR = \frac{\cos D - \cos(\phi - \phi') \cdot \cos(\psi - \psi')}{\sin(\phi - \phi') \sin(\psi - \psi')},$$

and in the triangle QAB

$$\cos AQB = -\frac{\cos \iota - \cos \phi' \cos \psi'}{\sin \phi' \sin \psi'};$$

$$\text{and } PQR + AQB = \pi ;$$

$$\begin{aligned} \therefore \cos D &= \cos (\phi - \phi') \cos (\psi - \psi') \\ &+ \frac{\sin (\phi - \phi') \sin (\psi - \psi')}{\sin \phi' \sin \psi'} (\cos \iota - \cos \phi' \cos \psi'), \end{aligned}$$

which determines D , ϕ' and ψ' being connected with ϕ and ψ by the equations

$$\sin \phi = \mu \sin \phi', \quad \sin \psi = \mu \sin \psi'.$$

81. If a pencil of light be refracted in any manner through a prism the directions of its axis before the first and after the second refractions are inclined at the same angle to the edge of the prism.

Let radii of a sphere be drawn parallel to the axis of the pencil at incidence on the first surface, after the first and after the second refraction, their directions being contrary to that of the pencil in each case, and let them meet the surface of the sphere in P , Q , R respectively (fig. 43). Also let radii parallel to the normals to the first and second surfaces and the edge of the prism meet the sphere in A , B , I . Draw the great circles AQP , BRQ , and join AB , IP , IR , IA , IB , by arcs of great circles. Then I is the pole of AB ,

$$IA = \frac{\pi}{2} = IB, \quad \sin AP = \mu \cdot \sin AQ, \quad \sin BR = \mu \cdot \sin BQ.$$

$$\begin{aligned} \therefore \cos IP &= \sin AP \cdot \cos QAI \\ &= \mu \sin AQ \sin QAB \\ &= \mu \sin BQ \sin QBA \\ &= \sin BR \cdot \cos QBI \\ &= \cos IR, \end{aligned}$$

$$\therefore IP = IR,$$

or the axis of the pencil before the first and after the second refraction is inclined at the same angle to the edge of the prism.

82. To determine the foci of a small pencil refracted obliquely through a prism, the axis of the pencil being perpendicular to the edge of the prism.

The pencil is supposed refracted at the edge of the prism in order to examine the effect of obliquity on the pencil independently of that of the thickness of the prism.

Let Q (fig. 44) be the origin of a small pencil whose axis is obliquely refracted through a prism in direction QAS . Let Q_1 , Q_2 be its primary and secondary foci after the first refraction, q_1 , q_2 those at emergence.

$$\text{Let } AQ = u, \quad Aq_1 = v_1, \quad Aq_2 = v_2,$$

ϕ , ϕ' the angles of incidence and refraction at the 1st surface,
 ψ , ψ' emergence and incidence 2d

From the first refraction

$$AQ_1 = \frac{\mu \cos^2 \phi'}{\cos^2 \phi} u.$$

Now the pencil emanating in the primary plane from Q_1 diverges in that plane after refraction from q_1 : hence if we suppose its course reversed, a pencil converging to q_1 will converge in the primary plane to Q_1 after refraction into the prism.

$$\therefore AQ_1 = \frac{\mu \cos^2 \psi'}{\cos^2 \psi} v_1.$$

$$\therefore v_1 = \frac{\cos^2 \phi'}{\cos^2 \phi} \cdot \frac{\cos^2 \psi}{\cos^2 \psi'} u \quad (1)$$

$$\text{So } AQ_2 = \mu u,$$

$$AQ_2 = \mu v_2,$$

$$\therefore v_2 = u \quad (2)$$

The angles ϕ' ψ' are known in terms of ϕ from the conditions

$$\sin \phi = \mu \sin \phi', \quad \sin \psi = \mu \sin \psi',$$

$$\phi' \pm \psi' = \text{refracting angle of prism.}$$

83. COR. 1. The primary or secondary focus of the emergent pencil is the nearer of the two to the prism as ϕ is less or greater than ψ .

84. COR. 2. If the pencil at emergence diverge from a point or $v_1 = v_2$,

$$\therefore \frac{\cos^2 \phi'}{\cos^2 \phi} \cdot \frac{\cos^2 \psi}{\cos^2 \psi'} = 1, \quad \phi = \psi,$$

which is also the condition of minimum deviation.

In this case the emergent pencil diverges from a point at the same distance as its origin from the edge of the prism, and the degree of divergence remains unaltered.

Combined refractions at spherical surfaces.

85. DEF. A lens is a portion of a refracting medium bounded by two surfaces of revolution which have a common axis called the axis of the lens.

OBS. The bounding surfaces of a lens will, unless the contrary be expressed, be considered spherical, under which denomination plane surfaces are included as a particular case, when the radius of the sphere is infinite.

86. Lenses have different names according to the nature of the surfaces.

In fig. 45.

a is a double convex lens,

b double concave,

c convexo-plane,

d concavo-plane,

e plano-convex,

f plano-concave,

g convexo-concave,

h concavo-convex.

Light is as usual supposed in these figures to come from the right, and the order of the terms which are combined to form the designation of any lens is that in which light is incident on the two surfaces.

Thus c and e are the same lens, but it is convexo-plane in the former case because light is incident first on the convex and then on the plane surface, plano-convex in the latter case because light is first incident on the plane and then on the convex surface.

OBS. A pencil is said to be refracted directly through a lens when the refraction at each surface is direct.

87. To find the geometrical focus of a pencil after direct refraction through a lens of which the thickness is neglected.

Let Q (fig. 46) be the origin of a pencil whose axis QAB passes directly through a lens of which the thickness AB may be neglected. Let F_1 and F be the geometrical foci of the pencil after one refraction and at emergence.

Let $AQ = u$, $BF = v$, r , s the radii of the first and second surfaces of the lens respectively, lines being regarded positive when measured in a direction contrary to that of the incident pencil.

From the refraction at the first surface,

$$\frac{\mu}{AF_1} - \frac{1}{u} = \frac{\mu - 1}{r}. \quad (34) \quad (1).$$

Now a pencil converging to F would have F_1 for its geometrical focus after refraction at B (61),

$$\therefore \frac{\mu}{BF_1} - \frac{1}{v} = \frac{\mu - 1}{s} \quad (2)$$

\therefore if the thickness be neglected or AF_1 considered = BF_1 ,

$$\frac{1}{v} - \frac{1}{u} = (\mu - 1) \left(\frac{1}{r} - \frac{1}{s} \right),$$

which determines the position of F .

88. COR. 1. If the thickness be sensible but small and $= t$, equation (2) becomes

$$\begin{aligned}\frac{\mu}{AF_1} - \frac{\mu t}{AF_1^2} - \frac{1}{v} &= \frac{\mu - 1}{s} \\\therefore \frac{1}{v} - \frac{1}{u} &= (\mu - 1) \left(\frac{1}{r} - \frac{1}{s} \right) - \frac{\mu t}{AF_1^2} \\&= (\mu - 1) \left(\frac{1}{r} - \frac{1}{s} \right) - \frac{t}{\mu} \left(\frac{1}{u} + \frac{\mu - 1}{r} \right)^2,\end{aligned}$$

or the effect of the thickness is to remove the point F farther from A by a distance

$$\frac{t}{\mu} \left(\frac{1}{u} + \frac{\mu - 1}{r} \right)^2 v^2.$$

89. DEF. The geometrical focus of a pencil of parallel rays refracted directly through a lens is called the principal focus of the lens, and the distance of this point from the surface of the lens is the focal length of the lens.

The focal length of a lens is generally denoted by the symbol f . Hence

$$\frac{1}{f} = (\mu - 1) \left(\frac{1}{r} - \frac{1}{s} \right).$$

90. When f is positive the lens is thinnest at its axis; when f is negative the lens is thickest at the axis. Thus lenses may be divided into two classes distinguished by the sign of the focal length. Those whose focal length is positive will be called concave lenses; those whose focal length is negative will be called convex lenses. Lenses may be constructed of an infinite variety of forms so as to have the same focal length.

91. A divergent pencil incident directly on a concave lens diverges after refraction. A convergent pencil consists at emergence of diverging, parallel, or converging rays as its point of convergence is at a distance from the lens greater than, equal to, or less than the focal length of the lens.

A divergent pencil incident directly on a convex lens consists at emergence of diverging, parallel, or converging rays as its origin is at a distance from the lens less than, equal to, or greater than the focal length of the lens. A convergent pencil converges after refraction.

DEF. The reciprocal of the focal length of a lens is sometimes called the power of the lens.

92. When a pencil is refracted directly through a lens to find the point where the direction of any ray after refraction cuts the axis.

Let Q (fig. 47) be the origin of a pencil whose axis QAB is refracted directly through a lens of inconsiderable thickness. Let $QRST$ be the course of any ray whose directions after one refraction and at emergence cut the axis in q_1 and q respectively.

Let $AQ = u$, $Aq = v'$, r , s the radii of the first and second surfaces of the lens, lines being accounted positive when measured in a direction contrary to that of the incident pencil. $AR = y$.

Now from the first refraction,

$$\frac{\mu}{AQ_1} - \frac{1}{u} = \frac{\mu - 1}{r} + \frac{\mu - 1}{\mu^2} \left(\frac{1}{r} - \frac{1}{u} \right)^2 \left(\frac{1}{r} - \frac{\mu + 1}{u} \right) \frac{y^2}{2}. \quad (34) \quad (1)$$

If we suppose the course of the pencil reversed a ray of a pencil converging to q after refraction at BS cuts the axis in q_1 . Hence neglecting the thickness of the lens so that $BS = AR = y$, we obtain

$$\frac{\mu}{AQ_1} - \frac{1}{v'} = \frac{\mu - 1}{s} + \frac{\mu - 1}{\mu^2} \left(\frac{1}{s} - \frac{1}{v'} \right)^2 \left(\frac{1}{s} - \frac{\mu + 1}{v'} \right) \frac{y^2}{2}, \quad (2)$$

$$\therefore \frac{1}{v'} - \frac{1}{u} = (\mu - 1) \left(\frac{1}{r} - \frac{1}{s} \right)$$

$$+ \frac{\mu - 1}{\mu^2} \left\{ \left(\frac{1}{r} - \frac{1}{u} \right)^2 \left(\frac{1}{r} - \frac{\mu + 1}{u} \right) - \left(\frac{1}{s} - \frac{1}{v'} \right)^2 \left(\frac{1}{s} - \frac{\mu + 1}{v'} \right) \right\} \frac{y^2}{2}.$$

In the coefficient of y^2 we may use for v' its first approximate value v given by the equation

$$\frac{1}{v} - \frac{1}{u} = (\mu - 1) \left(\frac{1}{r} - \frac{1}{s} \right). \quad (87)$$

$$\therefore \frac{1}{v'} - \frac{1}{u} = (\mu - 1) \left(\frac{1}{r} - \frac{1}{s} \right)$$

$$+ \frac{\mu - 1}{\mu^2} \left\{ \left(\frac{1}{r} - \frac{1}{u} \right)^2 \left(\frac{1}{r} - \frac{\mu + 1}{u} \right) - \left(\frac{1}{s} - \frac{1}{v} \right)^2 \left(\frac{1}{s} - \frac{\mu + 1}{v} \right) \right\} \frac{y^2}{2},$$

which determines the position of the point q .

CoR. Aberration of the ray qS

$$= - \frac{\mu - 1}{\mu^2} \left\{ \left(\frac{1}{r} - \frac{1}{u} \right)^2 \left(\frac{1}{r} - \frac{\mu + 1}{u} \right) - \left(\frac{1}{s} - \frac{1}{v} \right)^2 \left(\frac{1}{s} - \frac{\mu + 1}{v} \right) \right\} \frac{v^2 y^2}{2}.$$

The position and magnitude of the least circle of aberration in this or any other case of combined direct refractions is given by (37).

93. To investigate the form of a lens of given focal length in order that the aberration of a given direct pencil may be the least possible.

The focal length of the lens being given the value of v for a proposed value of u is known. Hence we must have

$$\left(\frac{1}{r} - \frac{1}{u} \right)^2 \left(\frac{1}{r} - \frac{\mu + 1}{u} \right) - \left(\frac{1}{s} - \frac{1}{v} \right)^2 \left(\frac{1}{s} - \frac{\mu + 1}{v} \right) = \text{a minimum},$$

$$\begin{aligned} \text{while } (\mu - 1) \left(\frac{1}{r} - \frac{1}{s} \right) &= \frac{1}{f} \left\{ \right. \\ &\left. \text{and } \frac{1}{v} - \frac{1}{u} = \frac{1}{f} \right\}. \end{aligned}$$

In consistence with these two conditions we may assume

$$\frac{1}{v} = \frac{\alpha + 1}{2f}, \quad \frac{1}{u} = \frac{\alpha - 1}{2f},$$

$$\frac{1}{r} = \frac{x+1}{2(\mu-1)f}, \quad \frac{1}{s} = \frac{x-1}{2(\mu-1)f}.$$

$$\begin{aligned} \therefore \left(\frac{1}{r} - \frac{1}{u}\right)^2 \left(\frac{1}{r} - \frac{\mu+1}{u}\right) \\ = \frac{1}{8f^3(\mu-1)^3} \{x - (\mu-1)\alpha + \mu\}^2 \{x - (\mu^2-1)\alpha + \mu^2\} \quad (1) \end{aligned}$$

Now we pass from $\left. \begin{array}{l} \frac{1}{r} \text{ to } -\frac{1}{s} \\ \frac{1}{u} \text{ to } -\frac{1}{v} \end{array} \right\}$ by changing the sign of $\left\{ \begin{array}{l} x \\ \alpha \end{array} \right.$,

$$\begin{aligned} \therefore - \left(\frac{1}{s} - \frac{1}{v}\right)^2 \left(\frac{1}{s} - \frac{\mu+1}{v}\right) \\ = \frac{1}{8f^3(\mu-1)^3} [-\{x - (\mu-1)\alpha\} + \mu]^2 [-\{x - (\mu^2-1)\alpha\} + \mu^2] \quad (2) \end{aligned}$$

In adding (1) and (2) terms of odd dimensions in x and α disappear and the sum

$$\begin{aligned} &= \frac{1}{4f^3(\mu-1)^3} [\mu^2 \{x - (\mu-1)\alpha\}^2 + 2\mu \{x - (\mu-1)\alpha\} \{x - (\mu^2-1)\alpha\} + \dots] \\ &= \frac{1}{4f^3(\mu-1)^3} \{\mu(\mu+2)x^2 - 4\mu(\mu^2-1)\alpha x + \text{terms independent of } x\}, \end{aligned}$$

therefore if the aberration be a minimum,

$$0 = 2\mu(\mu+2)x - 4\mu(\mu^2-1)\alpha;$$

$$\therefore x = \frac{2(\mu^2-1)}{\mu+2}\alpha,$$

which determines r and s and consequently the form of the lens.

$$\text{The ratio of the radii of the lens} = \frac{s}{r} = \frac{x+1}{x-1}.$$

COR. If the incident pencil consist of parallel rays, or u be infinite and $\alpha = 1$, $\frac{s}{r} = \frac{\mu(2\mu+1)}{2\mu^2-\mu-4}$, a result which may be more readily obtained by the following independent process.

94. To investigate the form of a lens of given focal length in order that the aberration of a given direct pencil of parallel rays may be the least possible.

The aberration of a pencil of parallel rays refracted directly through a lens the radii of whose surfaces are r and s

$$\propto \frac{1}{r^3} - \left(\frac{1}{s} - \frac{1}{f} \right)^2 \left(\frac{1}{s} - \frac{\mu + 1}{f} \right),$$

$$\text{where } (\mu - 1) \left(\frac{1}{r} - \frac{1}{s} \right) = \frac{1}{f}, \text{ a given quantity.} \quad (A)$$

The former expression is therefore to be made a minimum consistently with the latter condition.

If we differentiate with respect to $\frac{1}{r}$ and observe that by equation (A) the differential coefficient of $\frac{1}{s} = 1$,

$$\begin{aligned} 0 &= \frac{3}{r^2} - \left(\frac{1}{s} - \frac{1}{f} \right) \left(\frac{3}{s} - \frac{2\mu + 3}{f} \right) \\ &= \frac{3}{(\mu - 1)^2} \left(\frac{1}{f} + \frac{\mu - 1}{s} \right)^2 - \left(\frac{1}{s} - \frac{1}{f} \right) \left(\frac{3}{s} - \frac{2\mu + 3}{f} \right); \\ \therefore \frac{s}{f} &= \frac{2(\mu - 1)(\mu + 2)}{2\mu^2 - \mu - 4}, \\ \therefore \frac{s}{r} &= 1 + \frac{1}{\mu - 1} \cdot \frac{s}{f} = \frac{\mu(2\mu + 1)}{2\mu^2 - \mu - 4}. \end{aligned}$$

95. Let OAB (fig. 48) be the axis of a lens, O , O' the centers of its first and second surfaces. Let R , S be points in these surfaces where the normals RO , SO' are parallel. Join SR , and produce it if necessary to cut the axis in C . Let r , s be the radii of the surfaces, $AB = t$.

By similar triangles $\frac{CO}{CO'} = \frac{r}{s}$,

$$\frac{r - AC}{s - t - AC} = \frac{r}{s},$$

$$AC = \frac{rt}{s - r}.$$

DEF. The point C whose position is thus dependent only on the form of the lens is called the Center of the Lens.

96. If a ray be refracted through the lens so that its direction between the two refractions passes through the point C , its directions at incidence and at emergence from the lens will be parallel to one another. For if R and S be the points of incidence and emergence, the surfaces at these points being parallel the case is the same as that of a ray refracted through a plate whose surfaces touch the lens at R and S .

When a pencil is refracted obliquely through a lens there will be an important difference produced according as the pencil is refracted centrally or excentrically, i. e. according as the direction of its axis between the two refractions does or does not pass through the center of the lens.

97. When a small pencil is obliquely and centrally refracted through a thin lens to find the distances of the foci of the emergent pencil from the center of the lens.

Let Q (fig. 49) be an origin of a small pencil whose axis $QCST$ is refracted obliquely and centrally through a lens, C being the point where the axis of the lens meets its first surface, which point if the thickness of the lens be neglected, is the center of the lens (95). Let Q_1 , Q_2 be the primary and secondary foci of the pencil after one refraction, q_1 , q_2 the primary and secondary foci of the emergent pencil. Let $CQ = u$, $Sq_1 = v_1$, $Sq_2 = v_2$, r , s the radii of the first and second surfaces of the lens, ϕ , ϕ' the angles of incidence and

refraction of the axis QC at C , and consequently the angles of emergence and incidence at S (96).

From refraction at the first surface

$$\frac{\mu \cos^2 \phi'}{CQ_1} - \frac{\cos^2 \phi}{u} = \frac{\mu \cos \phi' - \cos \phi}{r} \quad (51).$$

Now the pencil emanating in the primary plane from Q_1 diverges in that plane after the second refraction from q_1 ; hence if we suppose its course reversed a pencil converging to q_1 will converge in the primary plane to Q_1 after refraction into the lens.

$$\therefore \frac{\mu \cos^2 \phi'}{CQ_1} - \frac{\cos^2 \phi}{v_1} = \frac{\mu \cos \phi' - \cos \phi}{s},$$

the points C and S being regarded as coincident.

$$\therefore \frac{1}{v_1} - \frac{1}{u} = \frac{\mu \cos \phi' - \cos \phi}{\cos^2 \phi} \left(\frac{1}{r} - \frac{1}{s} \right) \quad (1).$$

$$\text{So } \frac{\mu}{CQ_2} - \frac{1}{u} = \frac{\mu \cos \phi' - \cos \phi}{r},$$

$$\frac{\mu}{CQ_2} - \frac{1}{v_2} = \frac{\mu \cos \phi' - \cos \phi}{s},$$

$$\therefore \frac{1}{v_2} - \frac{1}{u} = (\mu \cos \phi' - \cos \phi) \left(\frac{1}{r} - \frac{1}{s} \right) \quad (2).$$

Equations (1) and (2) determine the distances of the foci of the emergent pencil from S or C .

COR. 1. The positions of the foci of the refracted pencil being thus known, the investigation of (53) gives the position and magnitude of the circle of least confusion.

98. COR. 2. If ϕ be so small that its square may be neglected,

$$\frac{1}{v_1} \text{ or } \frac{1}{v_2} = \frac{1}{u} + (\mu - 1) \left(\frac{1}{r} - \frac{1}{s} \right).$$

Hence a pencil refracted centrically through a lens at small obliquity approximately converges to, or diverges from, a point at the same distance from the lens as the geometrical focus of a direct pencil with an origin at the same distance.

99. If the obliquity of the pencil be small, the positions of the foci of a pencil refracted centrically through a lens are conveniently determined by the distances from the center of their projections on the axis of the lens.

Let Q (fig. 50) be the origin of a small pencil whose axis is obliquely and centrically refracted through a thin lens of which C is the center, q_1 , q_2 the primary and secondary foci of the emergent pencil.

Draw QM , q_1m_1 , q_2m_2 perpendicular to the axis of the lens. Let $CM = h$, $Cm_1 = k_1$, $Cm_2 = k_2$, r , s the radii of the surfaces of the lens, ϕ , ϕ' the angles of incidence and the refraction of the axis of the pencil, the cubes of which may be neglected; $\therefore \phi' = \frac{\phi}{\mu}$.

$$CQ = h \sec \phi, \quad Cq_1 = k_1 \sec \phi, \quad Cq_2 = k_2 \sec \phi.$$

\therefore from equation (1) of (97)

$$\begin{aligned} \frac{1}{k_1} - \frac{1}{h} &= \frac{\mu \cos \phi' - \cos \phi}{\cos^3 \phi} \left(\frac{1}{r} - \frac{1}{s} \right) \\ &= (\mu - 1) \frac{1 + \frac{\phi^2}{2\mu}}{1 - \frac{3\phi^2}{2}} \left(\frac{1}{r} - \frac{1}{s} \right) \\ &= \frac{1}{f} \left\{ 1 + \left(3 + \frac{1}{\mu} \right) \frac{\phi^2}{2} \right\}. \end{aligned}$$

Let q be the first approximate position of q_1 or q_2 when ϕ^2 is neglected, and let qm perpendicular to the axis $= x$,

$$Cm = \kappa; \quad \therefore \frac{x}{\kappa} = \tan \phi = \phi;$$

$$\therefore \frac{1}{h_1} - \frac{1}{h} = \frac{1}{f} + \left(3 + \frac{1}{\mu}\right) \frac{x^2}{2f\kappa^2}. \quad (1)$$

$$\begin{aligned} \text{So } \frac{1}{h_2} - \frac{1}{h} &= \frac{\mu \cos \phi' - \cos \phi}{\cos \phi} \left(\frac{1}{r} - \frac{1}{s}\right) \\ &= \frac{1}{f} + \left(1 + \frac{1}{\mu}\right) \frac{x^2}{2f\kappa^2}. \end{aligned} \quad (2)$$

Equations (1) and (2) determine the positions of m_1 and m_2 .

COR. If $h = -f$, the pencil consists after refraction of rays very nearly parallel.

100. To find the geometrical focus of a pencil of rays after direct refraction through a sphere.

Let p be the distance from the center of a refracting sphere of the origin of a pencil whose axis is refracted directly through the sphere, q_1 , q the distance of the geometrical foci of the pencil from the center after refraction at the first and second surfaces respectively, lines being considered positive when measured from the center in a direction contrary to that of the incident pencil; r the radius of the sphere. Then from refraction at the first surface

$$\frac{1}{q_1} - \frac{\mu}{p} = -\frac{\mu - 1}{r} \quad (33). \quad (1)$$

and from refraction at the second surface, if the course of the pencil be supposed reversed (61),

$$\frac{1}{q_1} - \frac{\mu}{q} = \frac{\mu - 1}{r}. \quad (2)$$

$$\therefore \frac{\mu}{q} - \frac{\mu}{p} = -2\frac{\mu - 1}{r}, \quad \text{or } \frac{1}{q} - \frac{1}{p} = -2\frac{\mu - 1}{\mu r},$$

which determines the position of the geometrical focus of the emergent pencil.

101. DEF. The focal length of a refracting sphere is the distance from the center of the geometrical focus of a pencil of parallel rays after direct refraction through the sphere.

Hence if f = the focal length,

$$\frac{1}{f} = -2 \frac{\mu - 1}{\mu r};$$

$$\text{and } \frac{1}{q} - \frac{1}{p} = \frac{1}{f}.$$

102. When a pencil is refracted directly through a sphere to find the point where the direction of a given ray after refraction cuts the axis of the pencil.

Let Q (fig. 51) be the origin of a pencil whose axis QAC is refracted directly through a sphere whose radius is r . Let $QRST$ be the course of a given ray meeting the surface of the sphere in R and S , Q_1 the point where its direction after the first refraction cuts the axis of the pencil.

Produce QR and TS to meet in L ; join CL , CR , and draw Cm perpendicular to RS . Let $CQ = p$, $CQ_1 = q_1$, q' = the distance from C of the point where ST the direction of the given ray at emergence cuts the axis, lines being considered positive when measured from C in a direction contrary to that of the incident pencil. Also let $CL = g$, the cube of which will be neglected.

$$\therefore Cm = CL \cdot \frac{\sin CRm}{\sin CRL} = CL \cdot \frac{\sin CRQ_1}{\sin CRQ} = \frac{g}{\mu}.$$

$$\text{Now } \mu = \frac{\sin CRQ}{\sin QCR} \cdot \frac{\sin Q_1CR}{\sin CRQ_1} = \frac{CQ}{RQ} \cdot \frac{RQ_1}{CQ_1};$$

$$\therefore RQ_1 \cdot CQ = \mu \cdot RQ \cdot CQ_1.$$

$$\text{But } RQ^2 = p^2 + r^2 - 2pr \cos \frac{AR}{r},$$

$$= (p - r)^2 + \frac{p}{r} \cdot AR^2 \text{ approximately,}$$

$$= (p - r)^2 + \frac{p}{r} \cdot \left(\frac{p - r}{p} \right)^2 g^2$$

from the triangles LCQ , RAQ which are approximately similar,

$$= (p - r)^2 \left\{ 1 + \frac{g^2}{pr} \right\}.$$

$$\therefore RQ = (p - r) \left\{ 1 + \frac{g^2}{2pr} \right\}$$

$$= p - r + \left(\frac{1}{r} - \frac{1}{p} \right) \frac{g^2}{2}.$$

$$\text{So } RQ_1^2 = (q_1 - r)^2 + \frac{q_1}{r} \cdot \left(\frac{q_1 - r}{q_1} \right)^2 \frac{g^2}{\mu^2}$$

$$RQ_1 = q_1 - r + \left(\frac{1}{r} - \frac{1}{q_1} \right) \frac{g^2}{2\mu^2};$$

$$\therefore \mu q_1 \left\{ p - r + \left(\frac{1}{r} - \frac{1}{p} \right) \frac{g^2}{2} \right\} = p \left\{ q_1 - r + \left(\frac{1}{r} - \frac{1}{q_1} \right) \frac{g^2}{2\mu^2} \right\},$$

$$\frac{\mu}{r} - \frac{\mu}{p} + \mu \left(\frac{1}{r} - \frac{1}{p} \right) \frac{g^2}{2pr} = \frac{1}{r} - \frac{1}{q_1} + \left(\frac{1}{r} - \frac{1}{q_1} \right) \frac{g^2}{2\mu^2 q_1 r}.$$

If in the coefficient of g^2 the approximate value of $\frac{1}{q_1}$ be employed which is given by the equation

$$\frac{1}{q_1} = \frac{\mu}{p} - \frac{\mu - 1}{r},$$

$$\frac{1}{q_1} - \frac{\mu}{p} = -\frac{\mu - 1}{r} - (\mu - 1) \left(\frac{1}{r} - \frac{1}{p} \right) \left(\frac{1}{p} + \frac{1}{\mu r} \right) \frac{g^2}{2r} \quad (1).$$

From the second refraction if the course of the ray be considered reversed,

$$\frac{1}{q_1} - \frac{\mu}{q'} = \frac{\mu - 1}{r} - (\mu - 1) \left(\frac{1}{r} + \frac{1}{q} \right) \left(\frac{1}{q} - \frac{1}{\mu r} \right) \frac{g^2}{2r} \quad (2),$$

$$\therefore \frac{1}{q'} - \frac{1}{p} = -2 \frac{\mu - 1}{\mu r}$$

$$- \frac{\mu - 1}{\mu} \left\{ \left(\frac{1}{r} - \frac{1}{p} \right) \left(\frac{1}{p} + \frac{1}{\mu r} \right) - \left(\frac{1}{r} + \frac{1}{q} \right) \left(\frac{1}{q} - \frac{1}{\mu r} \right) \right\} \frac{g^2}{2r},$$

q the first approximate value of q' obtained by neglecting g^2 being used in the coefficient of g^2 .

$$\text{COR.} \quad \frac{1}{q} - \frac{1}{p} = -2 \frac{\mu - 1}{\mu r}.$$

Therefore aberration of the ray ST

$$= q' - q = \frac{\mu - 1}{\mu} \left\{ \left(\frac{1}{r} - \frac{1}{p} \right) \left(\frac{1}{p} + \frac{1}{\mu r} \right) - \left(\frac{1}{r} + \frac{1}{q} \right) \left(\frac{1}{q} - \frac{1}{\mu r} \right) \right\} \frac{g^2 q^2}{2r}.$$

This expression is rendered more convenient for computation by assuming in consistence with the equation

$$\frac{1}{q} - \frac{1}{p} = \frac{1}{f}$$

$$\frac{1}{q} = \frac{a + 1}{2f}, \quad \frac{1}{p} = \frac{a - 1}{2f};$$

$$\therefore \text{aberration} = - \left\{ a^2 - \frac{\mu}{(\mu - 1)^2} \right\} \frac{g^2}{2f(a + 1)^2}.$$

Combined Lenses.

103. To find the geometrical focus of a pencil of rays after direct refraction through a series of lenses in contact whose axes are coincident.

Let Q (fig. 52) be the origin of a pencil whose axis $QC_1C_2\dots$ is refracted directly through a series of n lenses with a common axis whose centers are C_1, C_2, \dots . Let u be the distance of

Q from C_1 , v_1, v_2, \dots the distances from C_1, C_2, \dots respectively of the geometrical foci of the pencil after refraction through the first, second, ... lenses; f_1, f_2, \dots the focal lengths of the successive lenses, lines being considered positive when measured in a direction contrary to that of the incident pencil; then if the thickness of each lens be neglected,

$$\left. \begin{aligned} \frac{1}{v_1} - \frac{1}{u} &= \frac{1}{f_1} \\ \frac{1}{v_2} - \frac{1}{v_1} &= \frac{1}{f_2} \\ &\dots\dots\dots \\ \frac{1}{v_n} - \frac{1}{v_{n-1}} &= \frac{1}{f_n} \end{aligned} \right\}$$

$$\therefore \frac{1}{v_n} - \frac{1}{u} = \Sigma \left(\frac{1}{f} \right)$$

which determines the position of the geometrical focus of the emergent pencil.

COR. If the lenses be separated by finite intervals $a_1, a_2, \dots a_{n-1}$, we have in place of the equations of the proposition

$$\left. \begin{aligned} \frac{1}{v_1} - \frac{1}{u} &= \frac{1}{f_1} \\ \frac{1}{v_2} - \frac{1}{v_1 + a_1} &= \frac{1}{f_2} \\ &\dots\dots\dots \\ \frac{1}{v_n} - \frac{1}{v_{n-1} + a_{n-1}} &= \frac{1}{f_n} \end{aligned} \right\}.$$

By eliminating $v_1, v_2, \dots v_{n-1}$ between these n equations v_n is determined.

104. When a pencil is refracted directly through a series of lenses with a common axis to find the point where the direction of a given ray at emergence cuts the axis.

Let f_1, f_2, \dots, f_n be the focal lengths of n lenses in contact whose axes are coincident, u the distance from the center of the first lens of the origin of a pencil directly refracted through the lenses, v_1, v_2, \dots the distances from the centers of the first, second, ... lenses of the geometrical foci of the pencil after refraction through these lenses respectively, $v_1', v_2' \dots$ the distances from the same centers of the points of intersection of a given ray with the axis after refraction through the successive lenses, lines being considered positive when measured in a direction contrary to that of the incident pencil. Then if the given ray be incident on the lenses supposed each indefinitely thin at a distance y from the axis

$$\frac{1}{v_1'} - \frac{1}{u} = \frac{1}{f_1} + \omega_1 y^2,$$

the coefficient of y^2 in (92) being denoted by ω_1 .

Now the given ray is refracted through the second lens in the same manner as if it were a ray of a pencil proceeding from an origin at distance v_1' from the center of that lens;

$$\therefore \frac{1}{v_2'} - \frac{1}{v_1'} = \frac{1}{f_2} + \omega_2 y^2,$$

.....

$$\frac{1}{v_n'} - \frac{1}{v_{n-1}'} = \frac{1}{f_n} + \omega_n y^2,$$

$$\therefore \frac{1}{v_n'} - \frac{1}{u} = \Sigma \left(\frac{1}{f} \right) + \Sigma (\omega) \cdot y^2,$$

which determines the point where the direction of the given ray at emergence cuts the axis.

$$\text{COR.} \quad \frac{1}{v_n'} - \frac{1}{u} = \Sigma \left(\frac{1}{f} \right) + \Sigma (\omega) y^2,$$

$$\frac{1}{v_n} - \frac{1}{u} = \Sigma \left(\frac{1}{f} \right);$$

$$\therefore \frac{v_n' - v_n}{v_n^2} = - \Sigma (\omega) \cdot y^2$$

therefore aberration of the given ray $= - \Sigma (\omega) v_n^2 y^2$

$$= - \frac{\Sigma (\omega)}{\left\{ \frac{1}{u} + \Sigma \left(\frac{1}{f} \right) \right\}^2} y^2.$$

OBS. The term $\Sigma (\omega)$ may be rendered more convenient for computation by substitutions similar to those of (93).

SECTION IV.

EXCENTRICAL REFLECTION AND REFRACTION, AND IMAGES.

105. OBLIQUE reflection or refraction has been defined to be excentrical when the point of incidence of the pencil is not a definite point of the reflecting or refracting surface called the center. In central reflection or refraction the direction of the reflected or refracted pencil has been defined by the angle of reflection or refraction of its axis, and its form by the distances of its foci from the point of incidence; in other cases it will be requisite to determine the projections of these foci on the axis of the surface, and to define the direction of the axis of the pencil by its inclination to the axis of the surface and the position of the point where it cuts this axis.

The laws of reflection and refraction and the calculations of the positions of the foci of a small pencil after oblique reflection or refraction would enable us to determine accurately the direction and form of a pencil excentrically reflected or refracted. The excentrical pencils however which occur in instruments have in general small inclinations to the axis of the reflecting or refracting surface, and therefore approximate computations of their direction and form are sufficient instead of the more laborious operations by which these elements would be exactly determined.

For these investigations the reader is referred to Coddington's Optics, Part I, but is advised to attend to uniformity of direction in the measuring of lines in cases where it is neglected in that work. Results will here be given with reference to the places where the processes which lead to them are found.

106. To find the course of the axis of a pencil after excentrical reflection at a spherical surface.

OBS. In the investigations of excentrical pencils the axis of the pencil is always supposed in the same plane with the axis of the reflecting or refracting surface, since this is the only case which generally occurs in optical instruments.

Let XA (fig. 53) be the axis of a pencil incident at A on a spherical reflecting surface of which O is the center of the surface and C the center of the face, AY the axis of the reflected pencil, CYX the axis of the reflecting surface.

Let $CO = r$, $CX = b$, $CY = c'$, lines being considered positive when measured from C in the direction more nearly opposite to that of the incident pencil. Also let $AC = y$, $AYC = \eta$, $AXC = \epsilon$.

$$\text{Then as in (26)} \quad \frac{XA}{YA} = \frac{XO}{YO},$$

whence if powers of y above the first be neglected, and c represent the first approximate value of c' , since then X , Y may be considered conjugate foci (28),

$$\frac{1}{c} + \frac{1}{b} = \frac{2}{r},$$

$$\text{and} \quad \frac{\tan \eta}{\tan \epsilon} = \frac{b}{c}.$$

107. But proceeding to a second approximation by retaining y^2 we have,

$$\frac{1}{c'} + \frac{1}{b} = \frac{2}{r} + \left(\frac{1}{r} - \frac{1}{b} \right)^2 \frac{y^2}{r},$$

$$\text{or if } \frac{1}{b} = \frac{1 + \beta}{r},$$

$$\frac{1}{c'} + \frac{1}{b} = \frac{2}{r} + \frac{\beta^2 y^2}{r^3}, \quad \text{Coddington 25.}$$

$$\text{and } \frac{\tan \eta}{\tan \epsilon} = \frac{b}{c} \left\{ 1 + \left(-\beta + \frac{\beta^2}{1 - \beta} \right) \frac{y^2}{r^2} \right\}.$$

108. To find the foci of a small pencil after excentric reflection at a spherical surface.

In the figure of the last proposition let Q be the origin of the pencil supposed small, q_1, q_2 the primary and secondary foci of the reflected pencil. Draw QM, q_1m_1, q_2m_2 perpendicular to the axis of the surface.

Let $CM = h, Cm_1 = k_1, Cm_2 = k_2$, lines being considered positive when measured from C in the direction more nearly opposite to that of the incident pencil. Also let $AC = y$.

If κ be the first approximate value of k_1 or k_2 given by the equation $\frac{1}{\kappa} + \frac{1}{h} = \frac{2}{r}$ and which may be used in the coefficient of y^2 for k_1 or k_2 ,

$$\frac{1}{k_1} + \frac{1}{h} = \frac{2}{r} + \left\{ \frac{1}{2h} \left(\frac{1}{b^2} - \frac{1}{hr} \right) + \frac{1}{2\kappa} \left(\frac{1}{c^2} - \frac{1}{\kappa r} \right) + \frac{1}{r} \left(\frac{1}{r} - \frac{1}{b} \right)^2 \right\} y^2,$$

$$\frac{1}{k_2} + \frac{1}{h} = \frac{2}{r} + \left\{ \frac{1}{2h} \left(\frac{1}{b^2} - \frac{1}{hr} \right) + \frac{1}{2\kappa} \left(\frac{1}{c^2} - \frac{1}{\kappa r} \right) - \frac{1}{r} \left(\frac{1}{r} - \frac{1}{b} \right)^2 \right\} y^2.$$

$$\text{Let } \frac{1}{h} = \frac{1+a}{r}, \quad \therefore \frac{1}{\kappa} = \frac{1-a}{r};$$

$$\frac{1}{b} = \frac{1+\beta}{r}, \quad \therefore \frac{1}{c} = \frac{1-\beta}{r};$$

and let \varkappa be the distance from the axis of the reflector of the first approximate position of q_1 or q_2 when y^2 is neglected,

$$\therefore \frac{1}{k_1} + \frac{1}{h} = \frac{2}{r} + (3V-1) \frac{\varkappa^2}{\kappa^2 r},$$

$$\frac{1}{k_2} + \frac{1}{h} = \frac{2}{r} + (V-1) \frac{\varkappa^2}{\kappa^2 r}.$$

Coddington 25.

COR. If the position of the circle of least confusion be considered as the bisection of $q_1 q_2$, (55) the distance of

its projection upon the axis from the first approximate position of m_1 or m_2 is $(2V - 1) \frac{x^2}{r}$ and the diameter of the circle

$$\frac{\lambda}{\kappa} \cdot \frac{Vx^2}{r}.$$

109. To find the course of the axis of a pencil after excentrical refraction through a thin lens.

Let b, c' be the distances from the center of the lens of points where the axis of the pencil cuts the axis of the lens before and after refraction, r, s the radii of the first and second surface of the lens, lines being considered positive when measured in the direction more nearly opposite to that of the incident pencil. Also let y be the distance of the point of incidence of the axis from the center of the lens, ϵ, η the inclinations between the axis of the lens and that of the pencil before and after refraction, f the focal length of the lens.

Then by an investigation similar to that of (87) if powers of y above the first be neglected, and c represent the first approximate value of c' ,

$$\frac{1}{c'} - \frac{1}{b} = (\mu - 1) \left(\frac{1}{r} - \frac{1}{s} \right),$$

$$\text{and } \frac{\tan \eta}{\tan \epsilon} = \frac{b}{c}.$$

110. But proceeding to a second approximation by retaining y^2 we have

$$\begin{aligned} \frac{1}{c'} - \frac{1}{b} &= (\mu - 1) \left(\frac{1}{r} - \frac{1}{s} \right) \\ &+ \frac{\mu - 1}{\mu^2} \left\{ \left(\frac{1}{r} - \frac{1}{b} \right)^2 \left(\frac{1}{r} - \frac{\mu + 1}{b} \right) - \left(\frac{1}{s} - \frac{1}{c} \right)^2 \left(\frac{1}{s} - \frac{\mu + 1}{c} \right) \right\} \frac{y^2}{2}. \end{aligned}$$

$$\text{Also } \frac{\tan \eta}{\tan \epsilon} = \frac{b}{c'} \left[1 + \frac{\mu - 1}{\mu} \left\{ \frac{1}{r} \left(\frac{1}{r} - \frac{1}{b} \right) - \frac{1}{s} \left(\frac{1}{s} - \frac{1}{c} \right) \right\} \frac{y^2}{2} \right]$$

Coddington 99.

$$\text{If } \frac{1}{b} = \frac{\beta - 1}{2f} \quad \text{and} \quad \therefore \quad \begin{cases} \frac{1}{c} = \frac{\beta + 1}{2f}, \\ \frac{1}{s} = \frac{x - 1}{2(\mu - 1)f} \end{cases}$$

$$\frac{1}{c'} - \frac{1}{b} = \frac{1}{f} + \frac{1}{\mu(\mu - 1)^2} \{(\mu + 2)x^2 - 4(\mu^2 - 1)\beta x + (\mu - 1)^2(3\mu + 2)\beta^2 + \mu^3\} \frac{y^2}{8f^3},$$

$$= \frac{1}{f} + B \frac{y^2}{8f^3} \text{ suppose.}$$

$$\text{and } \frac{\tan \eta}{\tan \epsilon} = \frac{b}{c'} \left[1 + \frac{1}{\mu(\mu - 1)} \{(\mu + 1)x - (\mu - 1)\beta\} \frac{y^2}{4f^2} \right]$$

$$= \frac{b}{c'} \left\{ 1 + E \cdot \frac{y^2}{4f^2} \right\} \text{ suppose,}$$

$$= \frac{b}{c} \left\{ 1 + \left(\frac{B}{\beta + 1} + E \right) \frac{y^2}{4f^2} \right\}.$$

111. To determine the foci of a small pencil after excentrical refraction through a lens.

If f be the focal length of the lens,

$\left. \begin{matrix} h \\ k_1 \\ k_2 \end{matrix} \right\}$ the distances from the center of $\left\{ \begin{matrix} \text{origin,} \\ \text{primary focus,} \\ \text{secondary focus.} \end{matrix} \right.$
the projections on the axis of

κ the first approximate value of k_1 or k_2 given by the equation $\frac{1}{\kappa} - \frac{1}{h} = \frac{1}{f},$

x the distances from the axis of the first approximate position of either focus; then

$$\frac{1}{k_1} - \frac{1}{h} = \frac{1}{f} + \left(3V + \frac{1}{\mu}\right) \frac{s^2}{2f\kappa^2},$$

$$\frac{1}{k_2} - \frac{1}{h} = \frac{1}{f} + \left(V + \frac{1}{\mu}\right) \frac{s^2}{2f\kappa^2}.$$

Coddington 102.

where V is a function of r, s, h, b .

112. To find the course of the axis of a pencil after excentrical refraction through a series of lenses which have a common axis.

Let the axis of the pencil cut the axis of the lenses before and after refraction through the first lens at distances b_1, c_1 from its center and let f_1 be the focal length of the lens. If the affixes 2, 3, ... denote similar quantities relatively to the second, third, ... lenses and if the thickness of the lenses be neglected, as a first approximation we have the system of equations

$$\frac{1}{c_1} - \frac{1}{b_1} = \frac{1}{f_1}, \quad \frac{1}{c_2} - \frac{1}{b_2} = \frac{1}{f_2}, \dots\dots\dots(109),$$

whence c_n is at last to be found.

Also if ϵ be the inclination of the axis of the pencil to that of the lenses before incidence, $\eta_1, \eta_2, \dots \eta_n$ its inclinations after refraction through the several lenses,

$$\frac{\tan \eta_1}{\tan \epsilon} = \frac{b_1}{c_1}, \quad \frac{\tan \eta_2}{\tan \eta_1} = \frac{b_2}{c_2}, \dots \frac{\tan \eta_n}{\tan \eta_{n-1}} = \frac{b_n}{c_n} \quad (109),$$

$$\therefore \frac{\tan \eta_n}{\tan \epsilon} = \frac{b_1 b_2 \dots b_n}{c_1 c_2 \dots c_n}.$$

113. The equations for the second approximation are

$$\frac{1}{c_1} - \frac{1}{b_1} = \frac{1}{f_1} + B_1 \frac{y_1^2}{8f_1^3}, \quad (110)$$

$$\frac{1}{c_2} - \frac{1}{b_2} = \frac{1}{f_2} + B_2 \frac{y_2^2}{8f_2^3},$$

$$\dots\dots\dots = \dots\dots\dots$$

$$\text{and } \frac{\tan \eta_n}{\tan \epsilon} = \frac{b_1 b_2 \dots b_n}{c_1 c_2 \dots c_n} \left\{ 1 + \Sigma \left(R \frac{y^2}{f^2} \right) \right\}.$$

Coddington 122.

114. DEF. A lens is equivalent to a system of lenses on the same axis when an excentrical pencil after refraction through it is inclined at the same angle to the axis as if it had been refracted through the system of lenses, the single lens having the position of that lens of the system on which the pencil is first incident.

If F be the focal length of a lens equivalent to the lenses in the last proposition, and if the pencil after refraction through it cut the axis at an angle η at a distance c , then using first approximations we have

$$\frac{\tan \eta}{\tan \epsilon} = \frac{b_1}{c}, \quad (109)$$

$$\frac{1}{c} - \frac{1}{b_1} = \frac{1}{F}, \quad (109)$$

$$\therefore \frac{b_1}{F} + 1 = \frac{\tan \eta}{\tan \epsilon} = \frac{\tan \eta_n}{\tan \epsilon} = \frac{b_1 b_2 \dots b_n}{c_1 c_2 \dots c_n}, \quad (112)$$

$$\frac{1}{F} = \frac{b_2 b_3 \dots b_n}{c_1 c_2 \dots c_n} - \frac{1}{b_1}.$$

If, as is generally the case in the eye glasses of telescopes, b_1 be very large and $\frac{1}{b_1}$ may be neglected,

$$F = \frac{c_1 c_2 \dots c_n}{b_2 b_3 \dots b_n}.$$

115. To find the foci of a small pencil after excentrical refraction through a series of lenses which have a common axis.

The distances from the center of the projections of the foci on the axis are given by equations similar to those of (111).

Coddington 124.

Images.

116. DEF. If a luminous body be placed before a reflecting or refracting surface, a pencil will emanate from each point in the surface of the body and will have after reflection or refraction a geometrical focus or circle of least confusion according as the incidence is direct or oblique. The curve or surface which is the locus of these foci or circles of least confusion is called the image of the luminous body.

117. DEF. If an image consist of points through which the light passes it is called *Real*; in other cases it is *Virtual*.

Hence a screen placed at any point of an image will receive illumination only when the image is real.

118. DEF. An image is erect when corresponding points of the object and image are on the same side of the axis of the reflecting or refracting surface.

119. The following is an instance of the formation of an image in the most simple manner, viz. by direct pencils.

The image formed by direct pencils reflected at a spherical surface of a circular arc concentric with the surface is another concentric circular arc.

Let O (fig. 54) be the center of the spherical reflecting surface, Q a point in the circular arc, QFO the axis of a direct pencil from Q of which after reflection F is the geometrical focus, and consequently the point of the image corresponding to the point Q of the object. If r be the radius of the sphere

$$\frac{1}{OF} = \frac{2}{r} - \frac{1}{OQ} \quad (31).$$

But OQ is invariable for all points in the object,

$\therefore OF$ image,

or the image is a circular arc, whose center is O .

COR. 1. The image is erect or inverted according as OF is positive or negative, i. e. as OQ is algebraically greater or less than $\frac{r}{2}$.

$$\text{COR. 2. } \frac{OQ}{OF} = \frac{2OQ - r}{r};$$

\therefore the image is magnified or diminished according as

$$2OQ - r \text{ is } < \text{ or } > r,$$

i. e. as $OQ < \text{ or } > \frac{r}{2}$ algebraically.

120. When the face of the reflecting or refracting medium is symmetrical about its center, as is generally the case, and when light emanates in all directions from each point of the object, then the pencil from any point by which the corresponding point of the image is formed is an oblique pencil whose axis passes through the center of the face of the surface or lens. The image in this case is formed by central pencils. If however the axis of the pencil from each point be compelled by any means, by a diaphragm for instance, to cut the axis of the surface or lens in a point which is not the center, the image is formed by excentric pencils.

In all cases of oblique reflection or refraction the image in general does not consist of an assemblage of points but of circles of confusion overlapping one another and therefore is called indistinct.

121. DEF. An image is said to be distorted when the distances from the axis of a point in the object and the corresponding point in the image are not in a constant ratio.

In images formed by direct reflection or refraction or by central refraction through a lens the direction of the axis of each pencil is unaltered and there is no distortion.

In images formed by excentric reflection or refraction the determination of the ratio $\frac{\tan \eta}{\tan \epsilon}$ (107, 110, 113) affords a means of estimating the degree of distortion. Let Q (fig. 55)

be a point in an object, of which an image is formed by excentric pencils confined to cut the axis AX in a given point X by a fixed diaphragm. Let q be the corresponding point in the image and Y the intersection of the axis of the reflected or refracted pencil with the axis of the surface or lens. Draw QM , qm perpendicular to the axis. Then if the inclinations of the axis of the pencil to that of the surface or lens be ϵ , η , and b , c' , the distances of X , Y from the surface,

$$qm = mY \tan \eta,$$

$$QM = MX \tan \epsilon.$$

$$\therefore \frac{qm}{QM} = \frac{mY}{MX} \cdot \frac{\tan \eta}{\tan \epsilon},$$

$$\propto (c' - Am) \frac{\tan \eta}{\tan \epsilon},$$

$$\propto \left(1 - \frac{Am}{c'}\right) (1 + Dy^2),$$

$$\propto 1 + Gy^2,$$

when substitution is made for $\frac{1}{c'}$ (107, 110).

G being a quantity dependent on the position of the diaphragm and object and on the form of the surface or lens.

This ratio is not constant in consequence of G being finite, the value of which may thus be used as a measure of the degree of distortion of the image.

In a similar manner the distortion can be found if the pencils be confined by a diaphragm to cut the axis in a given point after reflection or refraction.

If the object be a triangle whose plane is perpendicular to the axis of the mirror or lens its image projected on any parallel plane will be such as is represented in fig. (56) or (57), according as the quantity G is positive or negative, the parts being disproportionately enlarged or contracted as they are farther from the axis.

122. To find the curvature at the vertex of the image of a straight line perpendicular to the axis of a lens formed by pencils refracted centrically through the lens.

Let QA (fig. 58) be a straight line perpendicular to CA the axis of a lens by which an image of it is formed. Let q_1, q_2 be the primary and secondary foci of a central pencil from the point Q of the straight line, o the circle of least confusion, and therefore the point in the image belonging to the point Q of the object. Draw q_1m_1, q_2m_2, om perpendicular to the axis. Let f be the focal length of the lens, $CA = h, Cm_1 = k_1, Cm_2 = k_2, om = x$.

$$\begin{aligned} \text{Then } \frac{1}{k_1} - \frac{1}{h} &= \frac{1}{f} + \left(3 + \frac{1}{\mu}\right) \frac{x^2}{2f\kappa^2} \\ \frac{1}{k_2} - \frac{1}{h} &= \frac{1}{f} + \left(1 + \frac{1}{\mu}\right) \frac{x^2}{2f\kappa^2} \end{aligned} \quad (99)$$

where $\kappa = Ca$ is the first approximate value of k_1 or k_2 given by the equation

$$\begin{aligned} \frac{1}{\kappa} - \frac{1}{h} &= \frac{1}{f}, \\ \therefore \frac{1}{k_1} + \frac{1}{k_2} &= \frac{2}{\kappa} + \left(2 + \frac{1}{\mu}\right) \frac{x^2}{f\kappa^2}; \\ \therefore \frac{1}{k_1} - \frac{1}{\kappa} &= \left(3 + \frac{1}{\mu}\right) \frac{x^2}{2f\kappa^2}, \\ am_1 &= \kappa - k_1 = \left(3 + \frac{1}{\mu}\right) \frac{x^2}{2f}, \\ am_2 &= \kappa - k_2 = \left(1 + \frac{1}{\mu}\right) \frac{x^2}{2f}. \end{aligned}$$

If o be regarded as the bisection of q_1q_2 , (55)

$$2am = am_1 + am_2 = \left(2 + \frac{1}{\mu}\right) \frac{x^2}{f};$$

∴ the radius of curvature at the vertex of the image

$$= \frac{1}{2} \text{limit} \frac{x^2}{am}$$

$$= \frac{1}{\frac{2}{f} + \frac{1}{\mu f}},$$

and the curvature of the image at its vertex is measured by

$$\frac{2}{f} + \frac{1}{\mu f}.$$

123. COR. If the image be formed by central pencils refracted through a system of thin lenses in contact, the curvature of the image at its vertex is measured by

$$\left(2 + \frac{1}{\mu}\right) \Sigma \left(\frac{1}{f}\right),$$

and consequently depends only on the power of the combination, and not on the forms of the lenses or the position of the object.

124. COR. 2. By a similar investigation employing the equations of (111), it may be shewn that when an image is formed by excentric pencils of a straight line perpendicular to the axis of the lens, the radius of curvature at the vertex of the image

$$= \left\{ \frac{2V}{f} + \frac{1}{\mu f} \right\}^{-1},$$

and the curvature at the vertex is measured by $\frac{2V}{f} + \frac{1}{\mu f}$.

The condition that the image may be flat is $0 = \frac{2V}{f} + \frac{1}{\mu f}$; the condition that it may be distinct, or $k_1 = k_2$, is $V = 0$. Hence distinctness and flatness are not attainable together in the image.

If the image is formed by excentrical refraction through a series of lenses, the curvature of the image at the vertex is measured by $2\Sigma\left(\frac{V}{f}\right) + \Sigma\left(\frac{1}{\mu f}\right)$.

125. To find the course of the axis of a pencil after refraction at small obliquity through several media separated by spherical surfaces whose centers are in the same straight line.*

Let the surfaces which bound one of the media meet their common axis in A, B (fig. 59). Let RS be the direction of the axis of the pencil in this medium, ST its direction at emergence, and let these directions be produced to meet the axis of the surfaces in P, Q respectively. Let μ, μ' be the indices of refraction of the media wherein are RS, ST ; r the radius of the surface BS .

Then if the obliquity of the pencil be small, P and Q may be considered an origin and geometrical focus of a pencil directly refracted at BS ;

$$\therefore \frac{\mu'}{\mu} \cdot \frac{1}{BQ} - \frac{1}{BP} = \left(\frac{\mu'}{\mu} - 1\right) \frac{1}{r} \quad (32, 67),$$

$$\text{or} \quad \frac{\mu'}{BQ} - \frac{\mu}{BP} = \frac{\mu' - \mu}{r}. \quad (A)$$

$$\text{Also} \quad \frac{SB}{BP} = \tan SPB = \frac{RA}{AP}, \text{ approximately,}$$

$$\text{or} \quad SB = AR + \frac{AR}{AP} \cdot AB. \quad (B)$$

Let AB be the n^{th} of the series of media, μ_{2n} its refractive index, μ_{2n+2} that of the next medium, r_{2n+1} the radius of BS ,

$$AB = t_{2n} \mu_{2n}, \quad AR = y_{2n-1}, \quad BS = y_{2n+1}, \quad BP = \frac{\mu_{2n}}{\beta_{2n}} y_{2n+1},$$

* Dioptrische Untersuchung von C. F. Gauss. Gottingen, 1841.

$$AP = \frac{\mu_{2n}}{\beta_{2n}} y_{2n-1}, \quad BQ = \frac{\mu_{2n+2}}{\beta_{2n+2}} y_{2n+1}.$$

$$\text{Also let } \frac{\mu_{2n+2} - \mu_{2n}}{r_{2n+1}} = \rho_{2n+1}.$$

Equations (A) and (B) by these transformations give

$$\left. \begin{aligned} \beta_{2n+2} &= \beta_{2n} + \rho_{2n+1} y_{2n+1} \\ y_{2n+1} &= y_{2n-1} + t_{2n} \beta_{2n} \end{aligned} \right\}.$$

We have thus the system of equations

$$\beta_2 = \beta_0 + \rho_1 y_1,$$

$$y_3 = y_1 + t_2 \beta_2,$$

$$\beta_4 = \beta_2 + \rho_3 y_3,$$

$$\dots = \dots\dots\dots$$

$$y_{2s+1} = y_{2s-1} + t_{2s} \beta_{2s},$$

$$\beta_{2s+2} = \beta_{2s} + \rho_{2s+1} y_{2s+1}.$$

Here β_0 and y_1 being given from the direction of the axis of the pencil at incidence on the first medium, we have to find y_{2s+1} and β_{2s+2} , which define its direction in the $s+1^{\text{th}}$ medium.

Now suppose that

$$y_{2s+1} = g y_1 + h \beta_0,$$

$$\beta_{2s+2} = k y_1 + l \beta_0;$$

$$\text{then } y_{2s+3} = (k t_{2s+2} + g) y_1 + (l t_{2s+2} + h) \beta_0$$

$$= m y_1 + n \beta_0 \text{ suppose,}$$

$$\beta_{2s+4} = (m \rho_{2s+3} + k) y_1 + (n \rho_{2s+3} + l) \beta_0.$$

Therefore in obtaining alternately the successive values of y and β , the coefficients of y_1 and β_0 are formed from the preceding ones by the same law as the numerator and denominator of the continued fraction where the quotients are $\rho_1, t_2, \rho_3, \dots, t_{2s}, \rho_{2s+1}, \dots$. Hence if there be $s + 1$ media, we have

$$y_{2s+1} = gy_1 + h\beta_0,$$

$$\beta_{2s+2} = ky_1 + l\beta_0,$$

where $\frac{g}{h}$ is the last but one and $\frac{k}{l}$ the last of the above series of continued fractions. By the nature of such fractions $gl - hk = 1$.

126. Let A, B be the points where their common axis meets the first and last of the system of surfaces; P, Q the points where that axis meets the directions of the axis of the pencil in the first and last media. Let a, b, p, q be the distances of A, B, P, Q from an assumed point O in the axis of the surfaces, these distances being considered positive when they are measured in the direction more nearly contrary to that of the incident light.

$$\text{Then } \frac{\beta_0}{\mu_0} = \frac{y_1}{AP} = \frac{y_1}{p-a}, \quad \frac{\beta_{2s+2}}{\mu_{2s+2}} = \frac{y_{2s+1}}{BQ} = \frac{y_{2s+1}}{q-b};$$

$$\left. \begin{aligned} \therefore y_{2s+1} &= gy_1 + h \frac{\mu_0 y_1}{p-a} \\ \frac{\mu_{2s+2} y_{2s+1}}{q-b} &= ky_1 + l \frac{\mu_0 y_1}{p-a} \end{aligned} \right\};$$

\therefore if μ, ν be written for μ_0, μ_{2s+2} , we obtain

$$k \cdot \frac{p-a}{\mu} \cdot \frac{q-b}{\nu} + l \cdot \frac{q-b}{\nu} - g \cdot \frac{p-a}{\mu} = h.$$

$$\text{Let } p-a = u + \mu\theta, \quad q-b = v + \nu\phi;$$

$$\therefore 0 = k \frac{uv}{\mu\nu} + \frac{v}{\nu} (k\theta + l) + \frac{u}{\mu} (k\phi - g) + l\phi - g\theta + k\theta\phi - h.$$

Since between the four assumed quantities u, v, θ, ϕ there exist as yet only two equations, we may also suppose

$$k\theta + l = g - k\phi = \lambda,$$

$$\text{and } l\phi - g\theta + k\theta\phi - h = 0.$$

If ϕ and θ be removed,

$$0 = lg - hk - \lambda^2;$$

$$\therefore \lambda = 1.$$

$$\text{Hence} \quad \frac{\nu}{v} - \frac{\mu}{u} = k, \quad (\text{C})$$

$$\text{where } u = p - a - \mu \frac{1-l}{k},$$

$$v = q - b - \nu \frac{g-1}{k}.$$

DEF. Points in the axis of the system of surfaces whose distances from the assumed point O are

$$a + \mu \frac{1-l}{k}, \quad b + \nu \frac{g-1}{k},$$

are called focal centers.

Hence equation (C) gives the relation between the distances from the focal centers of the points where the axis of the surfaces meets the axis of the pencil in the first and last media.

127. COR. By comparison of equation (C) with (32), it is seen that the former is the same as would have been obtained if the pencil had at once been refracted from the first into the last medium through a spherical surface whose radius is $\frac{\nu - \mu}{k}$.

If $\mu = \nu$, equation (C) is the same as would have resulted if the pencil had been refracted through a thin lens of focal length $\frac{\mu}{k}$.

128. Let the directions of the axis of the pencil in the first and last media meet the axis of the surfaces in P , Q respectively, and perpendiculars to that axis through the focal centers M , N in S , T . (fig. 59*).

$$\text{Then } SM = \frac{\beta_0}{\mu} \cdot PM = \frac{\beta_0}{\mu} \left(p - a - \mu \frac{1-l}{k} \right) = y_1 - \beta_0 \frac{1-l}{k},$$

$$\begin{aligned} TN &= \frac{\beta_{2s+2}}{\nu} \cdot QN = \frac{\beta_{2s+2}}{\nu} \left(q - b - \nu \frac{g-1}{k} \right) \\ &= y_{2s+1} - \frac{g-1}{k} \beta_{2s+2} \\ &= gy_1 + h\beta_0 - \frac{g-1}{k} (ky_1 + l\beta_0) \\ &= y_1 - \beta_0 \frac{1-l}{k}; \end{aligned}$$

$$\therefore SM = TN.$$

Combining this result with that of (127) we see that if μ and ν are unequal, the path of the axis of the pencil in the last medium is the same as if the distance between the focal centers had been annihilated, and the first and last media separated by a spherical surface of radius $\frac{\nu - \mu}{k}$ at the point where the focal centers coincide. If μ , ν be equal, the path of the pencil in the last medium is the same as if a thin lens of that medium of focal length $\frac{\mu}{k}$ had been placed where the focal centers are supposed to coincide.

129. If

P_1 be the plane of P when Q is infinitely distant, $OP_1 = p_1$,

Q_1 Q P $OQ_1 = q_1$,

and if m , n be the distances of the focal centers M and N from O ,

$$\frac{\nu}{q_1 - n} = k, \quad \frac{\mu}{p_1 - m} = -k;$$

$$\therefore \frac{q_1 - n}{q - n} + \frac{p_1 - m}{p - m} = 1,$$

$$(p - p_1)(q_1 - q) = (m - p_1)(q_1 - n),$$

$$\text{or } PP_1 \cdot QQ_1 = MP_1 \cdot NQ_1,$$

P_1P , Q_1Q being measured in contrary directions from P_1 , Q_1 .

Hence the following construction results.

Let the direction of the axis of the pencil in the first medium, and a straight line parallel to it through P_1 , meet in S , H a straight line through M perpendicular to the axis of the surfaces (fig. 59*). Let straight lines through S and H parallel to the axis of the surfaces meet in T and G straight lines perpendicular to the same axis through N and Q_1 . Join TG by a straight line cutting the axis in Q : this is the direction of the axis of the pencil in the last medium.

$$\text{For } \frac{PP_1}{SH} = \frac{MP_1}{HM}, \quad \frac{QQ_1}{MH} = \frac{NQ_1}{SH};$$

$$\therefore PP_1 \cdot QQ_1 = MP_1 \cdot NQ_1.$$

130. By equation (C) the distances MP , NQ vanish together, or if the direction of the axis of the pencil in the first medium passes through M , its direction in the last medium passes through N . Let these directions be inclined to the axis at the small angles ϵ , η respectively;

$$\therefore \nu\eta = \beta_{2s+2} = ky_1 + l\beta_0 = ky_1 + l\mu\epsilon;$$

$$\text{but } ky_1 = \mu\epsilon(1 - l);$$

$$\therefore \nu\eta = \mu\epsilon.$$

Hence if a small object at P subtend the angle ϵ at M , its image will be at Q , and will subtend the angle η at N .

The linear magnitudes of the image and object have the ratio $\frac{\mu \cdot NQ}{\nu \cdot MP}$, and the image is erect or inverted as this fraction is positive or negative.

131. To find the focal centers of a lens.

In this case we may suppose the pencil to be refracted through three media, where the first and third are the same. Hence in the last proposition $\mu = \nu$; therefore $\eta = \epsilon$, or the axis of the pencil between its refractions at the surfaces of the lens passes through the center of the lens (96). If t be the thickness of the lens, r and s its radii, m , n the distances of M , N from the surfaces of the lens,

$$\mu \cdot \frac{s - r}{rt} - \frac{1}{m} = \frac{\mu - 1}{r},$$

$$\frac{1}{m} = \mu \cdot \frac{s - r}{rt} - \frac{\mu - 1}{r};$$

$$\therefore m = \frac{rt}{\mu(s - r - t) + t}.$$

$$\text{So } n = \frac{st}{\mu(s - r - t) - t}.$$

The distance between the focal centers $= \frac{\mu - 1}{\mu} \cdot t$, if t be so small that its square may be neglected.

The learner is advised to commit to memory the following formulæ.

$$1. \quad \text{Direct illumination} \propto \frac{\text{cosine } \angle \text{ of incidence}}{(\text{distance})^2}.$$

2. Direct refraction at a spherical surface.

$$\frac{\mu}{v'} - \frac{1}{u} = \frac{\mu - 1}{r} + \frac{\mu - 1}{\mu^2} \left(\frac{1}{r} - \frac{1}{u} \right)^2 \left(\frac{1}{r} - \frac{\mu + 1}{u} \right) \frac{y^2}{2}.$$

3. Oblique refraction at a spherical surface.

$$\left. \begin{array}{l} \frac{\mu \cos^2 \phi'}{v_1} - \frac{\cos^2 \phi}{u} \\ \frac{\mu}{v_2} - \frac{1}{u} \end{array} \right\} = \frac{\mu \cos \phi' - \cos \phi}{r}.$$

4. Excentrical refraction through a lens.

$$\left. \begin{array}{l} \frac{1}{k_1} - \frac{1}{h} = \frac{1}{f} + \left(3V + \frac{1}{\mu} \right) \frac{x^2}{2f\kappa^2} \\ \frac{1}{k_2} - \frac{1}{h} = \frac{1}{f} + \left(V + \frac{1}{\mu} \right) \frac{x^2}{2f\kappa^2} \end{array} \right\}.$$

Formulæ of refraction will be found to include the corresponding formulæ of reflection if μ be made $= -1$. Hence (2) and (3) include the formulæ for direct and oblique reflection at a spherical surface. If in the same r be infinite the formulæ for refraction at a plane surface result. Formula (4) gives that for central refraction by putting $V=1$, and for reflection by putting $\mu = -1$. The formula for direct refraction through a lens may be conveniently remembered by its connection in form with (2).

SECTION V.

ON THE DISPERSION OF LIGHT.

132. PENCILS of light have hitherto been considered homogeneous. The following experiment of Newton shews that a pencil of sunlight has not this uniform nature, but admits of decomposition into a system of pencils in each of which the rays have a peculiar degree of refrangibility.

133. If the light of the sun be admitted into a darkened room through a small aperture A (fig. 60) in a window shutter, the pencil of light after entering the room may be regarded as approximately a cone with A for its vertex and the sun's apparent diameter for its vertical angle. If this pencil be allowed to fall perpendicularly on a screen, a circular bright spot B of white light is visible. Let the pencil be now refracted upwards through a prism of glass or any other refracting medium placed very near to the aperture, the axis of the pencil passing perpendicularly to the edge of the prism C which is horizontal, and very near to it. If the pencil be now received perpendicularly on a screen an elongated stripe or spectrum D is seen. On turning the prism about its edge this spectrum first descends and then ascends, and when it is stationary on a very small angular motion of the prism in either direction, so that the prism is in a position of minimum deviation, and if DC is made equal to BC , the spectrum was found of the same horizontal breadth as the circular spot B but about five times longer and of different colours in different parts, being red at the lowest or least refracted end, then orange, yellow, green, blue, indigo, violet in succession as we proceed to the upper extremity.

Now since the axis of the pencil passes with minimum deviation perpendicularly to the edge of the prism, if light were homogeneous the refracted pencil would be a cone diverging from an origin at a distance $= CA$ from C and having the sun's apparent diameter for its vertical angle (84). This cone being received perpendicularly on a screen at a distance $CD = CB$, the appearance would be a circular spot exactly equal to B . The experiment therefore leads to the following theory :

Sun light consists of different species of light of all degrees of refrangibility within certain limits and of all varieties of colour. The red rays are the least and the violet rays the most refrangible. These colours are separated by refraction through a refracting substance.

OBS. The elongation of the spectrum in this experiment is that which was observed by Newton with the prism used by him. Its amount in different cases depends on the refracting angle of the prism and its material.

134. At any point of the spectrum formed in the manner which has been described the light consists of a mixture of more refrangible rays from one point of the sun's disc with less refrangible rays from a lower point, and consequently in this spectrum the rays of different refrangibility are not accurately separated. In order to avoid this mixture of lights of different refrangibility Wollaston and Fraunhofer admitted the sun's light through a very narrow slit at a considerable distance from the prism and parallel to its edge. The latter observer viewed the spectrum by a telescope, the former by the naked eye placed close to the prism.

Let A (fig. 61) be a section of the slit made by a plane perpendicular to the edge of the prism, C the object glass, c the eye piece of the telescope. Then if the prism be in such a position that the deviation is a minimum, all the rays of a given refrangibility after refraction through the prism will diverge from some point E at the same distance as A from the prism, and after refraction at the lens C will converge to e , which may be viewed through the eye piece c .

Less refrangible pencils will diverge from points towards *R* after refraction through the prism, and will converge to points towards *r* after refraction through the lens. In like manner pencils of more refrangible light after being refracted through the prism and lens will converge to points towards *v*. Hence in the spectrum thus obtained the light at any point consists of rays of definite refrangibility perfectly free from admixture with light of a different refrangibility.

If the telescope in the above experiment be furnished with a micrometer the position of any fixed line in the spectrum can be accurately determined.

135. In observing the spectrum formed in this manner by the light of the sun Fraunhofer discovered that it was interrupted by nearly 600 lines, the strongest of which subtend in breadth an angle of from 5'' to 10''. The positions of a few of the most remarkable of these lines are indicated in fig. 62. *A* is a well defined line a little within the red end of the spectrum. At *a* a group of several lines form a band. *B* is a well defined line and of sensible breadth. In the space between *B* and *C* there are 9 very fine lines; *C* is a very dark line. Between *C* and *D* 30 very fine lines may be counted. At *D* in the orange are two strong lines separated by an extremely small interval. Between *D* and *E* about 84 lines may be distinguished. *E* lies in the green; it consists of several lines of which the middle line is rather broader than the others, so close that they appear to form one dark line. On both sides of *E* are other groups of fine lines much resembling *E* but not quite so dark. Between *E* and *b* are about 24 lines. At *b* are 3 strong lines of which the two furthest from *E* are very close. These are the strongest lines in the bright part of the spectrum. Between *b* and *F* about 50 lines may be counted. *F* is a strong line at the commencement of the blue. Between *F* and *G* may be counted about 185 lines variously grouped and of various breadth. *G* is a strong line in the indigo in the middle of a band of very fine lines. Between *G* and *H* are about 190 lines of various size. *H* is a strong line in the violet in the middle of a band of fine lines. Near it but farther from

G a similar band is seen. From *H* to *I* the end of the spectrum the lines are equally numerous.

Two of the fixed lines, probably *E* and *F*, had been discovered by Wollaston previously to the experiments of Fraunhofer.

As long as the source of light remains the same these lines occur in the same order and in the same colours whatever be the substance of which the prism is formed, their relative distances only varying. Fraunhofer ascertained by measuring with extreme accuracy the deviations of the lines *B*, *C*, *D*, *E*, *F*, *G*, *H* through prisms of the same substance with different refracting angles that for a ray corresponding to any one of the fixed lines the ratio of the sine of the angle of incidence to the sine of the angle of refraction was invariable. These ratios or the indices of refraction out of air into water at $15^{\circ}R$ are

$$\mu_B = 1.33095$$

$$\mu_C = 1.33171$$

$$\mu_D = 1.33357$$

$$\mu_E = 1.33585$$

$$\mu_F = 1.33780$$

$$\mu_G = 1.34127$$

$$\mu_H = 1.34417.$$

When the sun is very near the horizon the blue end of the spectrum disappears and lines are seen in the remaining part which were not before visible. Analogous effects are produced by interposing certain coloured vapours between the slit *A* and the source of light. In the spectrum of various fixed stars lines were observed differently situated from those in the solar spectrum. When the flame of a candle or oil lamp is placed behind *A* the spectrum is not interrupted by dark lines; a bright double line however is seen exactly occupying the place of the double line *D*. When the flame of a spirit lamp is placed behind *A* the double line *D* is extremely bright compared with the rest

of the spectrum. The spectrum formed by electrical light consists almost entirely of a few bright lines, some of which, according to the observations of Prof. Wheatstone, appear to depend upon the nature of the substances between which the spark is produced.

136. If we suppose light to be propagated by the undulations of an elastic medium, the lengths of the waves corresponding to the principal fixed lines will be as follows.

Extreme red	.00075 millimetres
<i>A</i>	·00074
<i>B</i>	·0006879
<i>C</i>	·0006559
<i>D</i>	·0005888
<i>E</i>	·0005265
<i>F</i>	·0004856
<i>G</i>	·0004296
<i>H</i>	·0003963
<i>I</i>	·00037
Extreme violet	.00036.

137. By measuring spectra in Fraunhofer's manner it is found that the distance between any the same two fixed lines has not a constant ratio to the distance between the extreme fixed lines where different media are used. This circumstance is called the Irrationality of Dispersion.

138. To determine the position of any given part of the spectrum seen through a prism.

Let Q (fig. 44) be the origin of a small pencil whose axis is obliquely refracted through a prism in direction QAS in a plane perpendicular to the edge of the prism.

Let q_1 be the primary focus of the emergent pencil, and

therefore to an eye in AS the given rays will appear to diverge from a line through q_1 . Then with the notation of (82)

$$\sin \phi = \mu \sin \phi', \quad \sin \psi = \mu \sin \psi',$$

$$\phi' + \psi' = i,$$

$$Aq_1 = \frac{\cos^2 \phi'}{\cos^2 \phi} \cdot \frac{\cos^2 \psi}{\cos^2 \psi'} AQ,$$

from which equations the place of q_1 may be determined.

$$\text{COR. } Aq_1 = \frac{\mu^2 - \sin^2 \phi}{1 - \sin^2 \phi} \cdot \frac{1 - \sin^2 \psi}{\mu^2 - \sin^2 \psi} AQ.$$

Now $\frac{\mu^2 - \sin^2 \phi}{1 - \sin^2 \phi}$ being an improper fraction is increased by an increase of ϕ , and $\frac{1 - \sin^2 \psi}{\mu^2 - \sin^2 \psi}$ being a proper fraction is increased by a diminution of ψ , which results from an increase of ϕ . Hence ϕ and Aq_1 increase or decrease together and therefore the distance of q_1 from the prism is greater, equal to, or less than that of Q as ϕ is greater, equal to, or less than ψ .

139. To find the angle between the axes of any two pencils of different refrangibility into which a pencil of white light incident on a prism with its axis perpendicular to the edge, is separated on emergence.

Let ι be the refracting angle of the prism, ϕ the angle of incidence of the axis of the pencil, ϕ' the angle of refraction of the axis of the pencil for which μ is the index of refraction, and ψ', ψ the angles of incidence and emergence of the same at the second surface, D the deviation of this axis. Then

$$\sin \phi = \mu \sin \phi', \quad \sin \psi = \mu \sin \psi',$$

$$\iota = \phi' + \psi',$$

$$D = \phi + \psi - \iota \dots \dots (77).$$

If $\mu + \delta\mu$ be the index of refraction for another of the pencils into which the incident pencil is separated by refraction, the deviation of its axis $= D + d_\mu D \cdot \delta\mu$ nearly, and the angle between the axes of this pencil and the former at emergence $= d_\mu D \cdot \delta\mu$.

Now ϕ is the same for both pencils, and ϕ' , ψ' , ψ are functions of μ ;

$$\therefore 0 = d_\mu \phi' + d_\mu \psi',$$

$$\cos \psi d_\mu \psi = \sin \psi' + \mu \cos \psi' d_\mu \psi',$$

$$0 = \sin \phi' + \mu \cos \phi' d_\mu \phi'.$$

$$\therefore \frac{\cos \psi}{\cos \psi'} d_\mu \psi = \frac{\sin \psi'}{\cos \psi'} + \frac{\sin \phi'}{\cos \phi'} = \frac{\sin \iota}{\cos \phi' \cos \psi'}.$$

$$\begin{aligned} \therefore \text{angle between the axes} &= d_\mu D \cdot \delta\mu \\ &= d_\mu \psi \cdot \delta\mu \\ &= \sec \phi' \sec \psi \sin \iota \cdot \delta\mu. \end{aligned}$$

COR. The variation of ψ is greater than the corresponding variation of ψ' (74) or ϕ' . Hence if the prism be turned round its edge from the position in which the spectrum is stationary so as to diminish ϕ , ψ increases faster than ϕ' decreases and the angle between the axes of the two pencils will continually increase: if the prism be turned in the contrary direction the same angle will decrease, and when ι is not too large, will obtain a minimum and afterwards increase.

140. To measure the refracting angle of a prism.

Let the prism be firmly fixed to a graduated circle provided with verniers, the edge of the prism being at the centre of the circle, perpendicular to its plane, and turned towards the object glass of a telescope fixed to the circle. Let the circle be turned until the image of a well defined distant object seen by reflection at one of the faces of the prism and

viewed by the telescope coincides with the intersection of the cross wires at the principal focus of the object glass of the telescope, and read the verniers of the circle. Turn the circle until the image of the same object seen by reflection in the other face of the prism coincides with the intersection of the wires, and read the verniers again. The difference of the readings or the angle through which the prism has been turned is equal to twice the angle of the prism.

141. To find the index of refraction of a ray corresponding to one of the dark lines out of air into any medium formed into a prism.

Let O (fig. 63.) be the center of a graduated circle moveable round an axis perpendicular to its own plane on a fixed circle carrying verniers, Cc a telescope the axis of the object glass of which passes through the axis of the circle and is parallel to the plane of the circle, A the intersection of the plane described by the axis of the telescope with the slit perpendicular to the plane of the circle by which light is admitted. The telescope is provided with cross wires at such a distance that when it is pointed to A the intersection of the wires may be at the image of A . Place the prism with its edge perpendicular to the plane of the circle so that its faces may be equidistant from the axis of the circle, and turn it until the incident and given emergent rays make equal angles with the normals to the faces of the prism or until the image of the given line is stationary. The prism retaining this position, turn the circle until the image of the line coincides with the intersection of the wires and read the verniers. Remove the prism and turn the circle until the intersection of the wires coincides with the image of A and read the verniers again. The difference of the two readings or the angle through which the circle has been turned between the two observations is the deviation of the given ray. If D be this deviation, ι the refracting angle of the prism, μ the index of refraction for the given line, then with the notation of (77),

$$D = 2\phi - \iota, \quad \iota = 2\phi'.$$

$$\therefore \mu = \frac{\sin \phi}{\sin \phi'} = \frac{\sin \frac{D + \iota}{2}}{\sin \frac{\iota}{2}}.$$

The refractive index of a fluid for a given line may be found in the same manner by enclosing the fluid in a hollow prism of glass, of which the sides are plates with their surfaces accurately parallel. The deviation of the axis of the pencil then arises entirely from refraction through the fluid.

OBS. When the prism is not sufficiently perfect to exhibit the fixed lines we must select by estimation the particular part of the spectrum for which the index of refraction is required, and measure the deviation as in the former case.

142. DEF. If μ_r , μ_v , μ be the indices of refraction for the extreme red and violet rays capable of producing a sensible impression on the eye of the observer and for rays of mean refrangibility out of air into any medium, the quantity $\frac{\mu_v - \mu_r}{\mu - 1}$ is called the dispersive power of the medium. This quantity is frequently denoted by the letter ϖ .

If the medium be formed into a prism with a refracting angle ι , and if D_r , D_v , D be the deviations for the extreme red and violet rays and for rays of mean refrangibility of the axis of a pencil which passes through the prism at a small angle of incidence and emergence, then

$$D_r = (\mu_r - 1) \iota, \quad (78)$$

$$D_v = (\mu_v - 1) \iota,$$

$$D = (\mu - 1) \iota.$$

$$\therefore \frac{D_v - D_r}{D} = \frac{\mu_v - \mu_r}{\mu - 1},$$

or the dispersive power of a medium formed into a thin prism is equal to the ratio of the angle between the axes of the extreme red and violet pencils to the deviation of the axis of the pencil of light of mean refrangibility.

Achromatic Combinations.

143. It is now to be considered in what manner prisms and lenses may be combined so as to refract a pencil with the least possible dispersion. Such a combination is called achromatic.

A pencil of sun light after refraction does not in general converge to or diverge from a point for two reasons; first from the unequal refrangibility of the different species of light of which it consists, and secondly from the finite breadth of the pencil and the curvature of the refracting surface. These causes of aberration being independent of one another may be separately considered. It will therefore be supposed for simplicity in investigating the conditions of achromatism that no spherical aberration exists in the pencils which we consider.

144. The possibility of an achromatic combination arises from the fact that the dispersion of a pencil and the deviation of its axis produced by a refracting medium are not proportional. If this were the case a combination which could destroy colour would be incapable of producing deviation, and consequently would be useless for the purpose for which such combinations are designed. But from dispersion not being proportional to deviation media can be found which produce the same dispersion in opposite directions of a given fixed line of the spectrum relatively to another, but different deviations in opposite directions in the axis of the pencil. If a pencil then be refracted through these media the two lines in question may be united, while the axis of the pencil suffers a deviation equal to the difference of the deviations which the media would separately produce.

If refracting media had no irrationality (137), then in providing a combination such that two given lines should not be separated, we should at the same time unite lights of all species. But since the colours are disproportionately dispersed in different media the other lines will in such a case be very nearly but not exactly united. A pencil therefore refracted through an achromatic combination will illuminate a screen with light still slightly coloured, and pro-

duce what is called a secondary spectrum. The fixed lines in this spectrum do not generally preserve the same order of succession which they have in the spectra which the media separately produce.

A combination of different media achromatic for all kinds of light being thus in general unattainable, it is customary to unite the most brilliant kinds of light of the spectrum, the rest remaining but partially united.

Achromatic Prisms.

145. A pencil of light passes through two prisms of small refracting angles, its axis passing perpendicularly to the edge of each, to find the condition of achromatism.

Let ι , ι' be the refracting angles of the prisms, μ , μ' the indices of refraction for a given line. Then the deviations which the prisms would separately produce for this line are $(\mu - 1)\iota$ and $(\mu' - 1)\iota'$ (78). Hence the total deviation for this line = $(\mu - 1)\iota + (\mu' - 1)\iota'$.

If $\mu + \delta\mu$, $\mu' + \delta\mu'$ be the indices of refraction for another fixed line, then if the combination be achromatic for these two lines the above deviation must be unaltered when μ , μ' are increased by $\delta\mu$, $\delta\mu'$ respectively.

$$\therefore 0 = \delta\mu \cdot \iota + \delta\mu' \cdot \iota'. \quad (A)$$

the condition of achromatism.

The ratio of ι to ι' being given by this equation for any pair of values of $\delta\mu$, $\delta\mu'$, that is for any two fixed lines of the spectrum, the values of ι and ι' can then be found so that the combination shall produce a given deviation in light of a given refrangibility, and unite the aforesaid fixed lines. These two lines only will in general be united, because from irrationality the ratio of $\delta\mu$ to $\delta\mu'$ is in general different for each pair of lines.

OBS. From equation (A) it is seen that the refracting angles ι , ι' have opposite signs, which indicates that the edges of the prisms are to be turned towards opposite parts.

146. A pencil of light passes through two prisms of the same substance, its axis passing perpendicularly to the edge of each, to find the condition of achromatism.

Let ι be the refracting angle of the first prism, μ its index of refraction for a given fixed line, ϕ , ϕ' the angles of incidence and refraction and ψ , ψ' the angles of emergence and incidence at the second surface of the axis of a pencil of light corresponding to this fixed line.

$$\begin{aligned}\therefore \sin \phi &= \mu \sin \phi', \quad \sin \psi = \mu \sin \psi', \\ \phi' + \psi' &= \iota. \quad (77)\end{aligned}$$

If $\mu + \delta\mu$ be the index of refraction for another fixed line, and $\phi' + \delta\phi'$, $\psi' + \delta\psi'$, $\psi + \delta\psi$ the values of ϕ' , ψ' , ψ for this line,

$$\begin{aligned}0 &= \delta\mu \sin \phi' + \mu \cos \phi' \delta\phi', \\ \cos \psi \delta\psi &= \delta\mu \sin \psi' + \mu \cos \psi' \delta\psi'. \\ 0 &= \delta\phi' + \delta\psi'. \\ \therefore \cos \psi \frac{\delta\psi}{\delta\mu} &= \sin \psi' + \frac{\cos \psi' \sin \phi'}{\cos \phi'}, \\ \cos \psi \cos \phi' \cdot \frac{\delta\psi}{\delta\mu} &= \sin (\psi' + \phi') = \sin \iota. \quad (1).\end{aligned}$$

Let the pencil now be refracted through a second prism of the same substance and let ι_1 be the refracting angle, ψ_1 , $\psi_1 + \delta\psi_1$ the angles of incidence of the axes of pencils corresponding to the two fixed lines before considered, ϕ'_1 , the angle of incidence on the second surface of the axis of the pencil for which the index is μ .

Then if the two pencils are parallel at emergence from this second prism

$$\cos \psi_1 \cos \phi'_1 \frac{\delta\psi_1}{\delta\mu} = \sin \iota_1. \quad (2).$$

$$\text{But } \delta\psi = \delta\psi_1,$$

$$\therefore \frac{\cos \psi \cos \phi'}{\sin \iota} = \frac{\cos \psi_1 \cos \phi'_1}{\sin \iota_1},$$

the condition of achromatism.

Achromatic Lenses.

147. In forming an achromatic combination of lenses the problem is different according as the refracted pencil passes centrically or excentrically. In the former case the pencils of different colours into which an incident pencil is separated have a common axis (96) in which their points of divergence or convergence (98) lie, and the combination is achromatized by making as many as possible of these points coincide. But in a pencil refracted excentrically the axes of the coloured pencils after refraction have different directions, so that there is an angular separation of their points of divergence or convergence, and the condition of achromatism will be that which will render the axes of as many as possible of these pencils parallel.

148. A pencil of light passes centrically with small obliquity through two thin lenses in contact, to find the condition of achromatism.

A central pencil whose obliquity is small converges to or diverges from a point at nearly the same distance from the center of the lens as a direct pencil. (98). It is thus sufficient to find the condition of achromatism of a direct pencil refracted through the two proposed lenses.

The relation between u, v the distances of the origin and geometrical focus of the pencil from the lenses is

$$\frac{1}{v} - \frac{1}{u} = \frac{1}{f_1} + \frac{1}{f_2}, \quad (103),$$

f_1 being the focal length of the first lens the radii of whose surfaces are r_1, s_1 , for a fixed line of the spectrum whose refractive index is μ_1 , similar quantities for the other lens being denoted by the affix 2,

$$\frac{1}{f_1} = (\mu_1 - 1) \left(\frac{1}{r_1} - \frac{1}{s_1} \right), \quad (89)$$

\therefore the alteration of $\frac{1}{f_1}$ for an increase $\delta\mu_1$ of μ_1 is

$$\delta\mu_1 \left(\frac{1}{r_1} - \frac{1}{s_1} \right);$$

and the alteration of $\frac{1}{f_2}$ for $\delta\mu_2$ the corresponding increase of μ_2 is $\delta\mu_2 \left(\frac{1}{r_2} - \frac{1}{s_2} \right)$. Hence if v be the same, or the combination achromatic for the two fixed lines to which these refractive indices belong,

$$\begin{aligned} 0 &= \delta\mu_1 \left(\frac{1}{r_1} - \frac{1}{s_1} \right) + \delta\mu_2 \left(\frac{1}{r_2} - \frac{1}{s_2} \right) \\ &= \frac{\delta\mu_1}{\mu_1 - 1} \cdot \frac{1}{f_1} + \frac{\delta\mu_2}{\mu_2 - 1} \cdot \frac{1}{f_2}, \end{aligned}$$

which is the condition of achromatism.

COR. If there be n lenses in contact, the condition of achromatism will be

$$0 = \Sigma \left\{ \delta\mu \left(\frac{1}{r} - \frac{1}{s} \right) \right\} = \Sigma \left(\frac{\delta\mu}{\mu - 1} \cdot \frac{1}{f} \right).$$

This equation can be satisfied for $n - 1$ systems of values of $\delta\mu$, so that the combination can unite n fixed lines of the spectrum. The ratios of the focal lengths being assigned by the condition of achromatism, the numerical values of the focal lengths can be determined, if the geometrical focus of the pencil for a given kind of light be given.

149. A pencil of parallel rays is refracted directly through two thin lenses separated by a given interval; to find the condition of achromatism.

If f_1, f_2 be the focal lengths of the lenses for a fixed line whose index in them is μ_1, μ_2 respectively, a the distance of their centers; the distance v from the second lens of

the principal focus of the combination for this kind of light is given by the equation

$$\frac{1}{v} = \frac{1}{f_1 + a} + \frac{1}{f_2} \quad (103. \text{ Cor.}).$$

Now if μ_1 become $\mu_1 + \delta\mu_1$,

$$\begin{aligned} \text{alteration of } \frac{1}{f_1 + a} & \text{ or } \frac{\frac{1}{f_1}}{1 + \frac{a}{f_1}} \\ &= \frac{\frac{1}{f_1} \left(1 + \frac{\delta\mu_1}{\mu_1 - 1} \right)}{1 + \frac{a}{f_1} + \frac{a}{f_1} \cdot \frac{\delta\mu_1}{\mu_1 - 1}} - \frac{\frac{1}{f_1}}{1 + \frac{a}{f_1}} \\ &= \frac{f_1}{(a + f_1)^2} \cdot \frac{\delta\mu_1}{\mu_1 - 1} - \frac{af_1}{(a + f_1)^3} \cdot \left(\frac{\delta\mu_1}{\mu_1 - 1} \right)^2, \end{aligned}$$

powers of $\frac{\delta\mu_1}{\mu_1 - 1}$ above the second being neglected;

and if μ_2 become $\mu_2 + \delta\mu_2$,

$$\text{alteration of } \frac{1}{f_2} = \frac{\delta\mu_2}{\mu_2 - 1} \cdot \frac{1}{f_2}.$$

Hence if v remain the same for the two pairs of refractive indices which we have considered, or if the combination be achromatic for the two corresponding kinds of light,

$$0 = \frac{f_1}{(a + f_1)^2} \cdot \frac{\delta\mu_1}{\mu_1 - 1} - \frac{af_1}{(a + f_1)^3} \left(\frac{\delta\mu_1}{\mu_1 - 1} \right)^2 + \frac{1}{f_2} \frac{\delta\mu_2}{\mu_2 - 1},$$

the condition of achromatism.

150. A pencil passes excentrically through two thin lenses separated by a given interval, its axis before incidence intersecting the common axis of the lenses in a given point; to find the condition of achromatism.

Let f_1, f_2 be the focal lengths of the lenses for a fixed line whose refractive index in them is μ_1, μ_2 respectively, a the distance between the centers of the lenses. Let the axis of the pencil of this kind of light before and after refraction through the first lens cut the axis of the lens at distances b_1, c_1 from the center of that lens, and before and after refraction through the second lens at distances b_2, c_2 from the centre of that lens; also let ϵ, η be its inclination to the axis of the lenses before refraction and at emergence.

Then if first approximations be used,

$$\frac{1}{c_1} - \frac{1}{b_1} = \frac{1}{f_1}, \quad (109)$$

$$\frac{1}{c_2} - \frac{1}{b_2} = \frac{1}{f_2},$$

$$b_2 = c_1 + a,$$

$$\frac{\tan \eta}{\tan \epsilon} = \frac{b_1 b_2}{c_1 c_2}.$$

$$\therefore \frac{b_1}{c_1} = 1 + \frac{b_1}{f_1},$$

$$\frac{b_2}{c_2} = 1 + \frac{c_1 + a}{f_2}.$$

$$\begin{aligned} \therefore \frac{\tan \eta}{\tan \epsilon} &= \frac{b_1 b_2}{c_1 c_2} = \frac{b_1}{c_1} + \frac{b_1}{f_2} + \frac{a b_1}{f_2 c_1} \\ &= 1 + \frac{b_1 + a}{f_2} + \frac{b_1}{f_1} + \frac{a b_1}{f_1 f_2}. \end{aligned}$$

$$\text{Now if } \left. \begin{matrix} \mu_1 \\ \mu_2 \end{matrix} \right\} \text{ become } \left\{ \begin{matrix} \mu_1 + \delta \mu_1 \\ \mu_2 + \delta \mu_2 \end{matrix} \right.,$$

$$\text{alteration of } \left. \begin{matrix} \frac{1}{f_1} \\ \frac{1}{f_2} \end{matrix} \right\} = \left\{ \begin{matrix} \frac{1}{f_1} \cdot \frac{\delta \mu_1}{\mu_1 - 1} \\ \frac{1}{f_2} \cdot \frac{\delta \mu_2}{\mu_2 - 1} \end{matrix} \right.$$

$$\begin{aligned} \dots\dots\dots \frac{1}{f_1 f_2} &= \frac{1}{f_1 f_2} \left\{ 1 + \frac{\delta \mu_1}{\mu_1 - 1} + \frac{\delta \mu_2}{\mu_2 - 1} \right\} - \frac{1}{f_1 f_2} \\ &= \frac{1}{f_1 f_2} \left\{ \frac{\delta \mu_1}{\mu_1 - 1} + \frac{\delta \mu_2}{\mu_2 - 1} \right\}, \end{aligned}$$

omitting the product of $\frac{\delta \mu_1}{\mu_1 - 1}$ and $\frac{\delta \mu_2}{\mu_2 - 1}$.

Hence if η be the same for the two pairs of refractive indices, or if the combination be achromatic for the two corresponding kinds of light,

$$0 = \frac{b_1 + a}{f_2} \cdot \frac{\delta \mu_2}{\mu_2 - 1} + \frac{b_1}{f_1} \cdot \frac{\delta \mu_1}{\mu_1 - 1} + \frac{a b_1}{f_1 f_2} \left\{ \frac{\delta \mu_1}{\mu_1 - 1} + \frac{\delta \mu_2}{\mu_2 - 1} \right\}, \quad (A)$$

the condition of achromatism.

151. COR. If the lenses be of the same substance or $\mu_1 = \mu_2$,

$$0 = \frac{b_1 + a}{f_2} + \frac{b_1}{f_1} + \frac{2 a b_1}{f_1 f_2},$$

$$0 = f_1 + f_2 + \frac{a f_1}{b_1} + 2 a.$$

$$\therefore a = - \frac{f_1 + f_2}{2 + \frac{f_1}{b_1}}.$$

If b_1 be very large we have approximately

$$a = - \frac{f_1 + f_2}{2}.$$

In this case condition (A) is satisfied for all corresponding values of $\frac{\delta \mu_1}{\mu_1 - 1}$ and $\frac{\delta \mu_2}{\mu_2 - 1}$, or the combination is achromatic for all kinds of light, as might have been foreseen since irrationality has here no existence.

152. If a pencil of compound rays be refracted directly through a thin lens to find the chromatic aberration.

Let C (fig. 64) be the center of a thin lens AB , through which a pencil is directly refracted, v , r the geometrical foci for the most and least refracted rays of the pencil, μ_v , μ_r the indices of refraction for the same respectively, μ the index of refraction for mean rays, v the distance from the center of the geometrical focus of these rays, u the distance from the center of the origin of the pencil, and r , s the radii of the lens, lines being accounted positive when measured from the center contrary to the direction of the incident pencil.

$$\text{Then } \frac{1}{Cv} - \frac{1}{u} = (\mu_v - 1) \left(\frac{1}{r} - \frac{1}{s} \right),$$

$$\frac{1}{Cr} - \frac{1}{u} = (\mu_r - 1) \left(\frac{1}{r} - \frac{1}{s} \right).$$

$$\therefore \frac{rv}{Cr \cdot Cv} = (\mu_v - \mu_r) \left(\frac{1}{r} - \frac{1}{s} \right).$$

Since rv is usually small compared with Cv or Cr , $Cv \cdot Cr = v^2$ nearly,

$$\begin{aligned} \text{and } (\mu_v - \mu_r) \left(\frac{1}{r} - \frac{1}{s} \right) &= \frac{\mu_v - \mu_r}{\mu - 1} \cdot (\mu - 1) \left(\frac{1}{r} - \frac{1}{s} \right) \\ &= \frac{\varpi}{f}, \end{aligned}$$

if ϖ be the dispersive power of the lens (142) and f its focal length for mean rays.

$$\therefore \text{chromatic aberration } rv = \frac{\varpi v^2}{f}.$$

COR. Of the two points v , r , v is the nearer to C , except when the origin of the pencil is between the center of the lens and the principal focus of rays incident in the contrary direction.

153. To find the magnitude of the circle of chromatic aberration of a pencil refracted directly through a thin lens.

If spherical aberration be neglected (143) v , r are the points of divergence or convergence of the most and least refracted rays. If then A , B be the extremities of the pencil in the plane ABv , and if Ar cut Bv produced in a , and Br cut Av produced in b , then AC being equal to BC , the circle whose diameter is ab and plane perpendicular to Cr is the smallest space through which the whole dispersed pencil passes, and is called the circle of chromatic aberration.

Let c the bisection of ab be the center of this circle, and let $CA = y = CB$.

If the pencil be not very large,

the triangles $\left. \begin{matrix} vab \\ rab \end{matrix} \right\}$ are similar to $\left\{ \begin{matrix} vAB \\ rAB \end{matrix} \right.$.

$$\therefore \frac{cv}{ab} = \frac{Cv}{AB}, \quad \frac{cr}{ab} = \frac{Cr}{AB}.$$

$$\therefore \frac{vr}{ab} = \frac{Cr + Cv}{AB} = \frac{v}{y} \text{ nearly,}$$

$$\therefore ab = \frac{y}{v} \cdot vr = \frac{\varpi yv}{f},$$

which determines the diameter of the circle.

SECTION VI.

ON VISION AND OPTICAL INSTRUMENTS.

154. DESCRIPTION of the Eye.

The human eye consists of transparent substances enclosed in two coats nearly spherical in form. Figure 65 represents a section of it by a plane through a line ACD called the axis of the eye, with respect to which the surfaces of the coats are symmetrical.

The exterior coat EDF called the Sclerotica is horny and opaque, except the front part A , which is transparent and slightly protuberant beyond the nearly spherical surface of the rest, and is called the Cornea. The second coat interior to this is called the Choroides; it is opaque but has a circular aperture GH behind the cornea called a Pupil, whose center is in the axis of the eye. A membrane called the Retina extends within the choroides over the back of the eye and has communication with the brain by nerves.

B is a solid transparent substance in the form of a double convex lens with AD for its axis fixed by tendons springing from the choroides. It is called the Crystalline. The spaces between the cornea and the crystalline, and the crystalline and retina are filled with transparent fluids called respectively the aqueous and vitreous humours. The refractive index of these humours out of air is nearly that of water; the refractive index of the crystalline is a little greater.

155. To explain the manner in which vision takes place.

Let the axis of the eye be directed to a point Q in a luminous object PQ . A pencil from any point P falls upon

and is refracted by the cornea. Of this pencil a portion limited by the aperture GH is again refracted by the aqueous humour, the crystalline, and the vitreous humour, and is made to converge very nearly to a point p on the retina. The impression thus made on the retina is communicated to the brain, and produces the sensation of vision of the point P .

156. The pupil is capable of voluntary expansion and contraction within the limits of about $\cdot 25$ and $\cdot 09$ inches, so as to admit a larger or smaller pencil of light as the object viewed is less or more brilliant. The eye is able to adapt itself to objects at different distances so as to make pencils of different degrees of divergency converge nearly to a point on the retina, but whether this is effected by an alteration of form of the cornea or the crystalline or of both is not sufficiently ascertained. This power of adaptation however does not enable the eye to see objects within a certain distance, in general about eight inches, but of an object sufficiently brilliant at any greater distance distinct vision can be obtained, unless the obliquity be so great that the point p does not fall on the retina. Convergent rays also incident on the eye can never be brought to convergence on the retina.

157. Defects of vision.

An eye which produces too great refraction of a pencil incident upon it brings pencils from distant points to convergence at points so far before the retina as to produce no distinct impression upon it. This defect is called short sight. On the other hand, for an eye which cannot sufficiently refract a pencil, the least distance of distinct vision is greater than eight inches, pencils from points within this least distance being brought to convergence behind the retina. This is long sight.

158. When an object is at such a distance as to be conveniently seen, a pencil from any point of it may from the smallness of the pencil be regarded as composed of parallel rays. Hence in optical instruments it is in general provided that a pencil by which vision is produced shall consist of parallel rays.

Vision through optical contrivances depends on the fact that if a pencil diverging from a given point fall on the eye it is immaterial whether that point be an actual source of light, or whether the rays have been made to converge to it and afterwards to diverge. An image therefore is visible in the same manner as a luminous object in the same position would be, with this limitation that from any point of a luminous object rays diverge in all directions, but from any point of an image rays diverge only in directions corresponding to the directions of those rays which form that point in the image.

159. In general explanations of vision through optical instruments spherical aberration may from its smallness be disregarded. Pencils may be considered after reflection or refraction to converge to or diverge from a point, and an excentrical pencil may be supposed to have the same point of divergence or convergence as the centrical pencil from the same origin.

160. Vision through a lens.

Let PQ (fig. 66, 67) be a small luminous object, C the center of a lens whose axis is CQ , E the center of the pupil of an eye whose axis coincides with the axis of the lens. A pencil of light diverging from a point P of PQ falls upon the lens, and after refraction may be considered as diverging from some point p in CP or CP produced or converging to some point p in PC produced (96). Thus pq an image of PQ is formed. If Eq be not less than the least distance of distinct vision of the eye, of the pencil diverging from p the pupil selects a portion $p rs$ which has been refracted excentrically through the lens, and by this the point p is visible. Thus the image pq will be seen by the eye E .

161. COR. If PQ be very near to the principal focus of the lens, so that the image pq may be very distant, the excentrical pencil pE by which the point p is seen may be considered to consist of rays parallel to pC .

162. DEF. When an object is seen through a lens the magnifying power of the lens is the quotient of the angle which the image seen subtends at the eye, divided by the angle which the object would subtend at the eye if it were in the position of the image and viewed directly.

163. To find the magnifying power of a lens for given positions with respect to the lens of the object and eye.

In the figures of the last proposition, the object being supposed very small, $\frac{pq}{Eq}$ is the angle which the image pq subtends at the eye. But the object PQ viewed at distance Eq by the naked eye would subtend the angle $\frac{PQ}{Eq}$. Hence if m denote the magnifying power of the lens

$$m = \frac{pq}{PQ} = \frac{Cq}{CQ} \text{ by similar triangles.}$$

Now if f be the focal length of the lens and if the thickness of the lens may be neglected

$$\begin{aligned} \frac{1}{Cq} - \frac{1}{CQ} &= \frac{1}{f}, \\ \therefore \frac{Cq}{CQ} &= 1 - \frac{Cq}{f}, \\ \therefore m &= 1 - \frac{Cq}{f}. \end{aligned}$$

164. To compute the circumstances of vision through a lens.

In the figures of the last proposition

(1) If f be positive, or the lens concave (90), Cq is positive or the image erect. Also Cq is less than f , therefore $m < 1$, or the image is diminished with respect to the object.

(2) If f be negative, or the lens convex, Cq is positive or negative, and the image erect or inverted, as CQ is less or greater than the focal length of the lens without reference to sign. In the former case $m > 1$, or the image is magnified: in the latter case the image is magnified if $\frac{Cq}{f} > 2$, or the distance of the object from the lens less than twice the focal length, otherwise the image is diminished.

165. A convex lens having the effect of producing an erect and more distant image of a near object assists the eye of a long sighted person, and a concave lens by producing an erect and nearer image of a distant object assists a near sighted person. This is the use of spectacles and eye glasses.

166. If an object be viewed through a convex lens, the object being not farther from the lens than its principal focus, the divergence of the pencil by which any point is seen is greater as the object is nearer to the lens.

167. To find by experiment the focal length of a lens.

If the lens be convex let C (fig. 68) be its center, and Q a small luminous object on its axis, of which an image is formed at q on the opposite side of the lens if CQ be greater numerically than the focal length of the lens. If f be the focal length,

$$\frac{1}{Cq} + \frac{1}{CQ} = -\frac{1}{f}. \quad (87).$$

Suppose the position of the lens such that Qq or $CQ + Cq$ is the least possible; then by differentiating the above equation with regard to CQ , the differential coefficient of Cq being -1 , we obtain

$$\frac{1}{Cq^2} - \frac{1}{CQ^2} = 0 \text{ or } Cq = CQ,$$

$$\therefore Qq = -4f.$$

Hence if the image of Q formed at q be received on a screen, and the lens and screen moved until the distance of the

image from Q is the least possible, the focal length of the lens without reference to algebraic sign is one fourth of this distance.

If the lens be concave let it be placed in contact with a convex lens whose focal length f' is such that the focal length F of the combination may be a negative quantity, the axes of the two lenses being coincident. Then if f' and F be determined in the preceding manner, f is determined from the formula

$$\frac{1}{F} = \frac{1}{f} + \frac{1}{f'}, \quad (103)$$

$$\text{or } \frac{1}{f} = \frac{1}{F} - \frac{1}{f'}.$$

Telescopes.

168. Extremely distant objects cannot be seen in consequence of the small pencil from any point of them which the pupil selects not having sufficient illuminating power to make a sensible impression on the retina; again the parts of many distant but visible objects cannot be distinguished because the distance between the parts subtends at the eye no appreciable angle. Now if an image of one of such objects be formed near the observer by a lens or reflector, and pencils converging to or diverging from points in this image be refracted through a lens to the eye, the condensation of light in the pencil from any point may be sufficient to render that point visible, and the directions of the axes of the pencils may include at the eye sensible angles. This is the principle of a telescope.

The manner in which the image of the object viewed is formed divides telescopes into two classes, refracting and reflecting telescopes, in the former of which the image is formed by a lens and in the latter by a reflector. These instruments we shall first describe in their simplest forms, deferring the explanation of the additions and modifications which improve the vision of objects through them.

169. *The Astronomical Telescope.* (Fig. 69).

ACB is a convex lens called the object glass fixed in a tube, and acb a convex lens called the eye glass fixed in another tube which slides in the former, the common axis of the tubes being the common axis of the lenses. The focal length of the object glass is numerically greater than that of the eye glass, and when the instrument is adjusted for viewing very distant objects, the distance between C, c the centers of the lenses is the sum of the focal lengths.

If the axis of the lenses be directed to the point Q of an object PQ which is so distant that a pencil incident on the object glass from any point of it may be considered to consist of parallel rays, the pencil from a point P after refraction through the object glass converges very nearly to a point p in PC produced (96), Cp being equal to the focal length of the object glass (98), and thus pq an inverted image of PQ is formed. This image is by construction at the principal focus of the eye glass, and therefore a pencil diverging from a point p of the image, consists after excentrical refraction through the eye glass of rays parallel to pc (161), and is fitted to render the point p visible to an eye applied to the eye glass (158). Thus an inverted image of PQ is seen through the telescope.

170. Field of view.

If P_1 be a point of the object such that the ray P_1A , of the pencil from it, is refracted in the line Ab which joins opposite parts of the object glass and eye glass, then every ray of this pencil, and also every ray of a pencil from any point nearer to Q than P_1 , falls upon the eye glass and is refracted to the eye. Again if P_2 be a point in the object such that the ray P_2B , of the pencil from it, is refracted in the line Bb which joins corresponding parts of the object glass and eye glass, then of this pencil this ray alone falls upon the eye glass and is refracted to the eye. Of a pencil from a point between P_1 and P_2 a portion reaches the eye, and is less as the point is more distant from Q , and no ray of a pencil from a point more distant than P_2 from Q is refracted by the eye glass. Hence on looking through the telescope points whose angular

distance from Q exceeds P_1CQ are more and more faint as this distance is greater, and points whose angular distance from Q exceeds P_2CQ are invisible. This gradual fading of objects at a distance from the center of the field is called a ragged edge of the field. It is remedied by placing a stop at the position of the image pq , so as to destroy that part of the image which would be seen by partial pencils. The angular extent of the uniformly bright field is then the angle subtended at C by the diameter of the opening of the stop.

171. *Galileo's Telescope.* (Fig. 70.)

ACB is a convex lens called the object glass fixed in a tube, and acb a concave lens called the eye glass fixed in another tube which slides in the former, the common axis of the tubes being the common axis of the lenses. The focal length of the object glass is numerically greater than that of the eye glass, and when the instrument is adjusted for viewing very distant objects the distance between C, c the centers of the lenses, is the difference of the focal lengths.

If the axis of the telescope be directed to a point Q of an object PQ which is so distant that a pencil incident on the object glass from any point of it may be considered to consist of parallel rays, the pencil from a point P after refraction through the object glass converges very nearly to a point p in PC produced, Cp being equal to the focal length of the object glass, and thus pq an inverted image of PQ would be formed. Of the pencil converging to any point p of this image the eye glass, which is about the size of the pupil of the eye, selects the portion prs which has been refracted excentrically at the object glass, and since pq is by construction at the principal focus of the eye glass, this portion of the pencil after central refraction through the eye glass consists of rays parallel to cp and is fitted to render the point p visible to an eye applied to the eye glass. Thus an erect image of PQ is seen through the telescope.

172. Field of view.

If P_1 be a point in the object such that the ray P_1A of a pencil from it is refracted in the line Aa which joins cor-

responding parts of the object glass and eye glass, then of the pencil from this point or from any point nearer to Q a portion falls upon the eye glass sufficient to fill it. Again if P_2 be a point in the object such that the ray P_2A of the pencil from it is refracted in the line Ab which joins opposite parts of the object glass and eye glass, then of this pencil this ray alone falls upon the eye glass and is refracted to the eye. Of a pencil from a point between P_1 and P_2 the portion which reaches the eye partially fills the eye glass and is less as the point is more distant from Q ; also no ray of a pencil from a point more distant from Q than P_2 is refracted by the eye glass. Hence there will be a ragged edge to the field of view which in this telescope is incurable, because the image formed by the object glass is virtual (117) and therefore cannot be limited by a stop.

173. DEF. When an object is seen by a telescope, the magnifying power of the telescope is the quotient of the angle which the image seen subtends at the eye divided by the angle which the object would subtend at the eye if viewed directly.

174. To find the magnifying power of the Astronomical Telescope or of Galileo's Telescope.

Since the point p of the image pq (fig. 69, 70) is seen by a pencil whose axis is parallel to pc , and the point q by a pencil whose axis is qc , the image of PQ subtends at the eye the angle pcq . But PQ in consequence of its distance would subtend at the eye if viewed directly the angle PCQ or pCq . Hence

$$\begin{aligned} \text{magnifying power} &= \frac{pcq}{pCq} \\ &= \frac{\tan pcq}{\tan pCq} \text{ approximately,} \\ &= \frac{Cq}{cq} \\ &= \frac{\text{focal length of object glass}}{\text{focal length of eye glass}}. \end{aligned}$$

175. *Newton's Telescope.* (Fig. 71.)

ACB is a concave spherical reflector whose center is O , and whose axis CO coincides with the axis of the tube at the extremity of which it is placed, DEF a small plane mirror inclined at 45° to the axis of the tube, acb a convex eye glass placed in a tube which slides in an aperture of the former tube, to the axis of which its axis is perpendicular.

If the axis CO be directed to a point Q of an object PQ which is so distant that a pencil incident on ACB from any point of it may be considered to consist of parallel rays, the pencil from a point P after reflection at this mirror converges very nearly to a point p' in PO produced, Op' being $= \frac{1}{2}CO$ (26 Cor. 1), and then $p'q'$ an inverted image of PQ would be formed. This pencil being reflected again by DF will converge very nearly to a point p , the length of path of any ray to p being equal to that to p' . (20). Thus pq an inverted image of PQ is formed, and the position of the eye piece is such that this image may be at its principal focus. Hence the pencil diverging from any point p of the image consists after excentrical refraction through the eye glass of rays parallel to pc , and is fitted to render the point p visible to an eye applied to the eye glass. Thus an inverted image of PQ is seen through the telescope.

176. *Field of view.*

If P_1 be a point in the object such that the ray of a pencil from it which, after reflection at ACB , is incident on the small mirror at D , is reflected by DF in Db the line joining opposite parts of the small mirror and eye glass, then of a pencil from P_1 , or from any point of the object nearer to Q than P_1 , every ray which falls on the small mirror is refracted by the eye glass to the eye. Again if P_2 be a point in the object such that the ray of a pencil from it which is incident on the small mirror at F , is reflected in Fb the line joining corresponding parts of the mirror and eye glass, then of the pencil from P_2 this single ray reaches the eye. Points between P_1 and P_2 appear more and more faint, and points at a greater distance than P_2 from Q are invisible.

177. To find the magnifying power of Newton's Telescope.

Let $p'q'$ (fig. 71) be the virtual image of PQ formed by the large mirror: pq is similar and equal to $p'q'$. Now pq subtends through the eye glass the angle $\frac{pq}{cq}$, and PQ subtends to the naked eye the angle POQ or $\frac{p'q'}{Oq'}$. Hence

$$\begin{aligned}\text{magnifying power} &= \frac{pq}{cq} \cdot \frac{Oq'}{p'q'} \\ &= \frac{Oq'}{cq} \\ &= \frac{\text{focal length of large reflector}}{\text{focal length of eye glass}}.\end{aligned}$$

178. *Herschel's Telescope.* (Fig. 72).

ACB is a concave spherical reflector whose center is O , and whose axis CO is inclined at a small angle to the axis of the tube at the extremity of which it is placed, acb a convex eye glass in a sliding tube attached to the interior of the larger tube, the axis of the eye glass cC being inclined to CO at the same angle as the axis of the larger tube.

If the axis of the larger tube be directed to a point Q of an object PQ which is so distant that a pencil incident on ACB from any point of it may be considered to consist of parallel rays, the pencil from a point P after reflection at the mirror ACB converges very nearly to a point p in PO produced, Cp being $= \frac{1}{2}CO$ (26 Cor. 1), and thus pq an inverted image of PQ is formed. The position of the eye glass is such that this image is at its principal focus, and therefore the pencil diverging from any point p of the image, consists after excentrical refraction through the eye glass of rays parallel to pc , and is fitted to render the point p visible to an eye applied to the eye glass. Thus an inverted image of PQ is seen through the telescope.

179. To find the magnifying power of Herschel's Telescope.

Since $pCO = PCO$, and $qCO = QCO$, (8)

$$\therefore pCq = PCQ.$$

Now pq viewed by the eye glass subtends the angle pcq , and PQ viewed by the naked eye subtends the angle PCQ or pCq .

$$\begin{aligned} \therefore \text{magnifying power} &= \frac{pcq}{pCq} \\ &= \frac{Cq}{cq} \\ &= \frac{\text{focal length of mirror}}{\text{focal length of eye glass}} \end{aligned}$$

OBS. The principle of Newton's and Herschel's telescope is the same, the plane reflector in the former having no object but to throw the image unaltered into another position where it may be more conveniently viewed. The mirror in Herschel's telescope is necessarily of large aperture, which is requisite for observing faint stars in order that a pencil sufficiently large to make them visible may reach the eye. The advantage of Herschel's construction over Newton's arises from no part of the incident pencils being stopped by the back of the small mirror, and no light being lost by the second reflection.

180. *Gregory's Telescope.* (Fig. 73).

ACB , DEF are two concave spherical reflectors with a common axis CE , which is the axis of a tube at one extremity of which ACB is placed, this mirror being much larger than DEF and of greater radius. The concavities of the mirrors are turned towards one another and the principal focus of ACB is between the center and principal focus of DEF . In a tube which is fixed in an aperture at the center of ACB is a convex eye glass acb , the axis of the eye glass coinciding with that of the reflectors.

If the axis of the reflectors be directed to a point Q in an object PQ which is so distant that a pencil incident on ACB from any point of it may be considered to consist of parallel rays, a pencil from any point P after reflection at ACB , converges very nearly to a point p in the straight line produced which joins P with the center of ACB , and thus pq an inverted image of PQ is formed at the principal focus of ACB . Since this image is between the center and principal focus of DEF , the pencil diverging from any point p after excentrical reflection at DEF converges to a point p' in the straight line produced which joins p with the center of DEF , and thus $p'q'$ an erect image of PQ is formed. The position of the eye glass is such that this image is at its principal focus, and therefore the pencil diverging from any point p' consists after excentrical refraction through the eye glass of rays parallel to $p'c$, and is fitted to render the point p' visible to an eye applied to the eye glass. Thus an erect image of PQ is seen through the telescope.

181. *Cassegrain's Telescope.* (Fig. 74).

ACB is a concave and DEF a convex spherical reflector with a common axis CE , which is the axis of a tube at one extremity of which ACB is placed. The mirror ACB , which is much larger and of greater radius than DEF , has its concavity turned towards the convexity of DEF , and the principal focus of ABC is between E and the principal focus of DEF . In a tube which is fixed in an aperture at the center of ACB is a convex eye glass acb , the axis of the eye glass being that of the reflectors.

If the axis of the reflectors be directed to a point Q of an object PQ which is so distant that a pencil incident on ACB from any point of it may be considered to consist of parallel rays, a pencil from any point P after reflection at ACB , converges very nearly to a point p in the straight line produced which joins P with the center of ACB , and then pq an inverted image of PQ would be formed at the principal focus of ACB . Since this image is between DEF and its principal focus, the pencil converging to any point

p after excentrical reflection at DEF converges to a point p' in the straight line produced which joins the center of DEF with p , and $p'q'$ an inverted image of PQ is formed. The position of the eye glass is such that this image is at its principal focus, and therefore the pencil diverging from any point p' consists after excentrical refraction through the eye glass of rays parallel to $p'c$, and is fitted to render the point p' visible to an eye applied to the eye glass. Thus an inverted image of PQ is seen through the telescope.

182. To find the magnifying power of Gregory's Telescope or Cassegrain's Telescope.

Let PC be the imaginary ray of the pencil from P (fig. 73, 74) which if the large mirror were uninterrupted, would be reflected at C its center, and let this ray be reflected at r by the small mirror in a direction whose intersection with the axis of the mirrors is Y , Y being nearly coincident with the principal focus of either mirror, because EC is large (106).

Now through the eye glass $p'q'$ subtends the angle $p'cq'$, and to the naked eye PQ subtends the angle PCQ or pCq ;

$$\begin{aligned}\therefore \text{magnifying power} &= \frac{p'cq'}{p'Yq'} \cdot \frac{rYq}{pCq} \\ &= \frac{Yq'}{q'c} \cdot \frac{EC}{EY} \text{ approximately,}\end{aligned}$$

or if f_0 , f_m , f_e be the focal lengths without reference to algebraic sign of the large mirror, small mirror, and eye glass respectively,

$$\text{magnifying power} = \frac{f_0}{f_e} \cdot \frac{f_0 \pm f_m}{f_m},$$

the upper sign belonging to Gregory's telescope and the lower sign to Cassegrain's.

183. Field of view in Gregory's and Cassegrain's Telescopes.

If the point P be such that the imaginary ray PC after reflection at the two mirrors just falls on the extremity a of the eye glass, then P is a limiting point of the field which is seen by at least half of each pencil reflected by the mirror ACB , and the angular extent of the field thus seen is double the angle PCQ . Hence with the construction and notation of (182)

$$\text{field of view} = 2PCQ$$

$$= 2 \frac{EY}{CE} \cdot aYC$$

$$= \frac{EY}{CE} \cdot \frac{acb}{cY}$$

$$= \frac{f_m}{f_0 \pm f_m} \cdot \frac{\text{aperture of eye glass}}{f_0 + Ce},$$

the upper sign belonging to Gregory's telescope and the lower sign to Cassegrain's.

184. The telescopes have here been described in their simplest forms for the purpose of explaining the principles of their construction. It remains to notice the defects of such instruments which render their modification by compound object glasses and eye glasses necessary, whereby these defects are diminished while the principle of the telescope is unaltered.

I. The Astronomical Telescope.

1. Let light be considered homogeneous. A pencil from any point of the object after oblique central refraction through a single object glass converges to two focal lines, and the image or assemblage of circles of confusion is indistinct and curved with its concavity to the object glass. Also a direct pencil is refracted with aberration. The compound object glass commonly used consists of lenses in

contact, and therefore by no arrangement of their forms can the indistinctness and curvature of the image be diminished (123). It therefore only remains to construct the lenses so as to produce the least possible aberration in a direct pencil of parallel rays.

If however a distinct and flat image were formed by the object glass, yet this image viewed through a single lens by excentrical pencils, would be indistinct, curved, and also distorted (120, 121). These defects are lessened by properly adjusting the forms of two or more lenses which form a compound eye piece. The three defects cannot be entirely removed together; each therefore is diminished as far as possible according to the artist's judgment and with reference to the use for which the telescope is intended.

2. Let light be considered as composed of different species, and let spherical aberration be disregarded (143). A pencil of such light refracted centrically through a simple object glass, is divided into pencils converging to a series of points in their common axis, and thus a series of coloured images differing slightly in position is formed. The most vivid of these images are united by a compound object glass of two lenses, the focal lengths of which are properly taken.

Again an achromatic image viewed by a single eye lens will from unequal refrangibility be confused, and the confusion will be of a worse kind than that produced by the object glass, because in the latter case the coloured pencils from the same point have a common axis, but in the present case when the refraction is excentrical they have not, and the coloured points corresponding to any point of the image formed by the object glass are spread over the field. To remedy this confusion the focal lengths of the lenses forming a compound eye piece are so adjusted that the axes of pencils of the most vivid colours belonging to the same point of the object emerge to the eye parallel to one another, in which case such pencils if they be small affect the eye as if they were coincident.

OBS. It is worthy of notice that the conditions of achromatism affect the focal lengths of the lenses combined in an

object glass or eye piece; the conditions of diminished indistinctness, curvature, and distortion of the image relate to the forms of the lenses.

II. Galileo's Telescope.

In the Astronomical telescope the refraction through the object glass is central, through the eye glass excentral, but in Galileo's the reverse is the case. Hence what has been said of the defects of a simple object glass and eye glass in an Astronomical apply respectively to the eye glass and object glass of a Galilean telescope. In this case the chromatic dispersion of the object glass is more unpleasant than that of the eye glass, and the object glass produces distortion in the image.

III. In the reflecting telescope there is spherical aberration from the curved reflectors, which produces indistinctness and curvature of the image, and in the telescopes of Gregory and Cassegrain, distortion. These defects are found to be lessened if the mirrors be not exactly spherical but figures generated by the revolution of Conic Sections. The defects of the single eye lens which are lessened by a compound eye piece are the same as have been mentioned in the Astronomical telescope.

185. In the Astronomical telescope since the pencils pass centrally through the object glass and excentrically through the eye glass, the field of view depends only on the aperture of the eye glass, the aperture of the object glass affecting only the brightness of the field. In Galileo's telescope on the contrary, where the refraction through the object glass is excentral, the field of view depends on its aperture, and this is the reason why this telescope is not so generally used for astronomical purposes as for a perspective or opera glass where small magnifying power is required. For with a high magnifying power and a field of any considerable extent the extreme pencils would be refracted by the object glass at such a distance from its axis as to make their chromatic dispersion unpleasant, and with difficulty diminished.

For the purposes for which this telescope is generally employed it has the convenience of exhibiting objects erect.

Object glasses.

186. The compound object glass commonly used in a refracting telescope consists of a lens of crown glass in contact with a lens of flint glass. The conditions which the combination has to fulfil are that it shall be achromatic for given kinds of light, and also aplanatic or free from spherical aberration.

Let affixes 1 and 2 denote symbols relative to the first and second lenses. Then if their thickness be neglected the condition of their achromatism for two species of light is

$$\frac{\delta\mu_1}{\mu_1 - 1} \cdot \frac{1}{f_1} + \frac{\delta\mu_2}{\mu_2 - 1} \cdot \frac{1}{f_2} = 0 \quad (148),$$

which determines the ratio of the focal lengths; and their absolute magnitudes can then be found from the focal length which the combination is intended to have. Probably the best object glasses unite the line *E* of the spectrum with that part of the spectrum between *D* and *C* where the intensity of the light is equal to that about *E*. The condition of achromatism leaves the forms of the lenses and the order of their position indeterminate.

If the object glass be aplanatic for a given species of light, then with the notation of (93),

$$\begin{aligned} 0 = & \frac{1}{\mu_1 (\mu_1 - 1)^2 f_1^3} \{ (\mu_1 + 2) x_1^2 - 4 (\mu_1^2 - 1) a_1 x_1 \\ & + (3\mu_1 + 2) (\mu_1 - 1)^2 a_1^2 + \mu_1^3 \} \\ & + \frac{1}{\mu_2 (\mu_2 - 1)^2 f_2^3} \{ (\mu_2 + 2) x_2^2 - 4 (\mu_2^2 - 1) a_2 x_2 \\ & + (3\mu_2 + 2) (\mu_2 - 1)^2 a_2^2 + \mu_2^3 \}, \\ \text{where } & \frac{a_1 + 1}{f_1} = \frac{a_2 - 1}{f_2}. \end{aligned}$$

If x_1 and x_2 be determined, the forms and succession of the lenses is defined.

Considering the rays of the pencil parallel at incidence on the object glass, we have

$$a_1 = 1, \quad a_2 = 1 + \frac{2f_2}{f_1};$$

and the above equation gives

$$0 = A_1 x_1^2 + B_1 x_1 + C_1 \\ + A_2 x_2^2 + B_2 x_2 + C_2,$$

when the coefficients are known functions of the refractive indices of the substances of the lenses, and the focal lengths already determined.

This equation may be satisfied by an unlimited number of pairs of values of x_1 and x_2 . If either of the lenses be taken of given form by assigning a particular value to x_1 or x_2 , this equation determines x_2 or x_1 , and so the form of the other lens.

- 187. Since this object glass then in addition to being aplanatic for parallel rays can be subjected to another condition, Sir J. Herschel has proposed as an advantageous addition that it shall also be made aplanatic for pencils diverging from any great but not infinite distance. This will be secured if the aberration vanish when $a_1 = 1 + \varepsilon$, ε being a small indeterminate quantity whose square may be neglected. On this substitution the aberration assumes the form $L + M\varepsilon$, and if this is to vanish for all small values of ε , $L = 0$, $M = 0$; or

$$0 = \frac{1}{\mu_1(\mu_1-1)^2 f_1^3} \{ (\mu_1+2)x_1^2 - 4(\mu_1^2-1)x_1 + (\mu_1-1)^2(3\mu_1+2) + \mu_1^3 \} \\ + \frac{1}{\mu_2(\mu_2-1)^2 f_2^3} \{ (\mu_2+2)x_2^2 - 4(\mu_2^2-1)x_2 \left(1 + 2\frac{f_2}{f_1} \right) \\ + (\mu_2-1)^2(3\mu_2+2) \left(1 + 2\frac{f_2}{f_1} \right)^2 + \mu_2^3 \},$$

$$\begin{aligned} \epsilon) = & \frac{1}{\mu_1(\mu_1-1)f_1^3} \{(\mu_1-1)(3\mu_1+2) - 2(\mu_1+1)x_1\} \\ & + \frac{1}{\mu_2(\mu_2-1)f_2^3} \{(\mu_2-1)(3\mu_2+2) \left(1 + 2\frac{f_2}{f_1}\right) \frac{f_2}{f_1} - 2(\mu_2+1)\frac{f_2}{f_1}x_2\}, \end{aligned}$$

whereby the lenses are determined in form and succession.

The forms of lenses computed by Precht from these formulæ were found to agree very closely with those adopted by Fraunhofer and his successors, some of which had been measured by Stampfer. (Precht, *Praktische Dioptric*, p. 92.) The first lens is of crown glass and double convex; the second of flint glass and concavo-concave, its first surface having very nearly the same curvature as that of the convex lens which it touches, the second surface having small curvature.

188. In these formulæ the lenses are supposed very thin, but in large object glasses the thickness of the lenses is too considerable to be neglected. In these cases the values of the radii of the lenses already obtained can only be considered as approximate results, and the following method of correcting them is proposed by Gauss*.

Let δx_1 , δx_2 be the required corrections to the values of x_1 and x_2 already obtained. These small alterations in x_1 and x_2 will produce an aberration whose value will be of the form $A.\delta x_1 + B.\delta x_2$, where A and B are to be found.

Let a numerical value of δx_1 be assumed, such however that its square may be neglected in comparison with the focal lengths of the lenses. Taking the lens corresponding to this assumed correction with the given thickness, let the path of a ray of light incident parallel to its axis be exactly computed as if it passed through a prism touching the lens at its points of incidence and emergence. The aberration of this ray from the approximate point of convergence already determined is thus known, and $= A.\delta x_1$, whereby A is determined. By giving a small arbitrary value to δx_2 , and by following

* *Lehrbuch der analytischen Optik* von I. C. E. Schmidt.

the same process with regard to the second lens, B is known. We have now the condition in order to the combination being aplanatic

$$0 = A \cdot \delta x_1 + B \cdot \delta x_2. \quad (1).$$

Again, neglecting spherical aberration, the points of convergence of two colours which are to be united will by reason of the alteration of the forms of the lenses be separated by a distance $A' \cdot \delta x_1 + B' \cdot \delta x_2$. If A' , B' be found by the method already described, then in order to the combination being achromatic for the two proposed colours

$$0 = A' \cdot \delta x_1 + B' \cdot \delta x_2. \quad (2).$$

Equations (1) and (2) give δx_1 and δx_2 , and therefore the corrected forms of the lenses.

189. Achromatic object glasses have also been constructed of three lenses in contact, consisting of a concave lens of flint glass between two convex lenses of crown glass. The conditions of achromatism can thus be satisfied for more species of light than in the former case; the forms of the lenses will be determined in a manner similar in principle to that which has been given above. Such object glasses are now not frequently employed in consequence of the difficulty of centering the lenses so that their axes may exactly coincide.

190. The compound eye pieces most generally used are those called the negative or Huygen's eye piece, and the positive or Ramsden's eye piece. Each consists of two convex lenses separated. The former is achromatic; the latter is designed to diminish the defects of indistinctness, curvature and distortion in the image seen.

Eye Pieces,

191. If in the expression for $\frac{\tan \eta}{\tan \epsilon}$ (110) we suppose $e=f$ approximately by considering b very large, the distortion produced by the lens $\propto \frac{1}{f^3}$. This circumstance led Huygen to expect a diminution of the distortion in an Astronomical

telescope by using two separated convex lenses for an eye piece in place of one, and dividing equally between them the deviation produced in an excentrical pencil. Using first approximations and the notation of (112) we then have

$$\frac{1}{c_1} = \frac{1}{f_1}, \text{ neglecting } \frac{1}{b_1} \text{ from its smallness,} \quad (1)$$

$$\frac{1}{c_2} - \frac{1}{b_2} = \frac{1}{f_2}, \quad (2)$$

and if a be the distance of the lenses,

$$b_2 = a + c_1, \quad (3)$$

$$\frac{\tan \eta_2}{\tan \eta_1} = \frac{b_2}{c_2}.$$

If from the distance of the object glass the axis of the pencil when incident on the first lens be considered parallel to the axis of the lenses, then the deviations produced in the axis by the first and second lenses are η_1 and $\eta_2 - \eta_1$. If these then be equal,

$$2\eta_1 = \eta_2,$$

or from the smallness of the angles,

$$2 \tan \eta_1 = \tan \eta_2.$$

$$\therefore b_2 = 2c_2,$$

$$\therefore b_2 = f_2 \text{ from (2),}$$

$$\text{and } a = -(f_1 - f_2) \text{ from (1) and (3).}$$

The construction adopted by Huygens in consistence with this condition was an eye glass of two convex lenses whose focal lengths are in the ratio of 3 to 1, the less powerful lens being placed first, and the distance between the lenses being the difference of the focal lengths.

It is a remarkable coincidence that if the lenses be of the same material this construction undesignedly fulfils the condition of achromatism of an excentrical pencil

$$a = -\frac{1}{2}(f_1 + f_2) \quad (151).$$

The Huygenian or negative eye piece therefore is achromatic.

192. Let ACB (fig. 75) be the object glass of an Astronomical telescope directed to a very distant object PQ , DEF the first lens or field glass, and acb the second lens or eye glass of Huygen's eye piece, the focal length of the former being three times that of the latter, and the distance Ec the difference and the semi-sum of the focal lengths without reference to sign. A pencil from a point P of the object after refraction through the object glass would converge very nearly to a point p , Cp being equal to the focal length of the object glass, but being excentrically refracted by the field glass converges to a point p' in Ep , and thus $p'q'$ an inverted image of PQ is formed. The position of the eye piece is such that q' is the bisection of Ec , or this image is at the principal focus of the eye glass. Hence a pencil from any point p' of the image after excentrical refraction at the eye glass consists of rays parallel to $p'c$, and is fitted to render the point p' visible to an eye applied to the eye glass. Thus an inverted image of PQ is seen through the telescope.

193. The compensation between the two lenses which renders this eye piece achromatic may be thus simply explained.

The deviation of the axis of a pencil of light produced by a convex lens is greater as the axis is refracted at a greater distance from the axis of the lens; for the axis of the pencil is refracted in the same degree as it would be by a prism whose surfaces touch the lens at the points where the axis of the pencil is incident and emergent, and therefore the deviation is greater as the refracting angle of such a prism is greater (78). Now when a pencil of light refracted by the object glass falls on the field glass, it is separated by it into a series of coloured pencils whose axes follow different courses, the deviation of the axis of the red pencil being least and of the violet greatest. The axes of the pencils do not cut the axis of the lenses between the lenses, and thus the axis of the red pencil falls on the eye glass at the greatest distance from the axis of the lenses and consequently is most refracted by it; the axis of the violet

pencil falling nearest to the axis of the eye glass is least refracted by it. Thus the pencils from the same point in the object which are least and most refracted by the field glass are respectively most and least refracted by the eye glass, and consequently may be parallel where they enter the eye.

COR. The position of the eye piece is determined by the condition of a direct pencil having the principal focus of the eye glass as its geometrical focus after refraction through the field glass. It must therefore at incidence on the field glass be converging to a point at a distance from that lens equal to half its focal length.

194. The focal lengths and positions of the lenses of the eye piece being determined, the forms of the lenses are chosen with reference to the use of the telescope. The conditions for reducing distortion, indistinctness, and curvature of the field being different, it depends on circumstances which of these defects is most to be avoided.

The functions G , V , (121, 111) being computed for each lens from the positions, focal lengths, and materials of the lenses, there will be no distortion if

$$0 = G_1 y_1^2 + G_2 y_2^2,$$

which gives

$$\begin{aligned} 0 = A_1 x_1^2 + B_1 x_1 + C_1 \\ + A_2 x_2^2 + B_2 x_2 + C_2, \end{aligned}$$

whence the form of one lens can be found corresponding to any assumed form of the other.

There will be distinctness if

$$\frac{V_1}{f_1} + \frac{V_2}{f_2} = 0,$$

which may be satisfied in a variety of ways.

The field is flat if

$$\frac{2V_1}{f_1} + \frac{1}{\mu_1 f_1} + \frac{2V_2}{f_2} + \frac{1}{\mu_2 f_2} = 0,$$

which cannot be satisfied by real values of x_1 and x_2 ; such values of these quantities are therefore to be used as reduce the former member of this equation most nearly to zero.

The best form on the whole for the lenses is that represented in the figure, the field glass being convexo-concave and the eye glass nearly convexo-plane.

195. Ramsden's eye piece, sometimes called the positive eye piece, is a combination of two separated convex lenses for the purpose of diminishing the effects of spherical aberration, which can be effected better by two lenses than one because more disposable quantities are thus introduced.

Let ACB (fig. 76) be the object glass of an Astronomical telescope directed to a very distant object PQ , DEF the first lens or field glass, and acb the second lens or eye glass of Ramsden's eye piece, these two lenses being of equal focal length and the distance Ec two thirds of the focal length of either. A pencil from a point P of the object after refraction through the object glass converges to a point p , Cp being equal to the focal length of the object glass, and pq an inverted image of PQ is formed at the principal focus of the object glass. The pencil from any point p of this image after excentrical refraction through the field glass, diverges from a point p' in Ep produced, and $p'q'$ a virtual inverted image of pq is formed, which from the position of the eye piece is at the principal focus of the eye glass. Therefore the pencil from any point p' after excentrical refraction through the eye glass consists of rays parallel to $p'c$, and is fitted to make the point p' visible to an eye applied to the eye glass. Thus an inverted image of PQ is seen through the telescope.

Obs. This eye piece is not achromatic and cannot be made so without sacrificing other advantages.

Cor. The position of the eye piece is determined by the condition of a direct pencil being so refracted by the field glass

as to have the principal focus of the eye glass for its geometrical focus. It must therefore at incidence on the field glass be diverging from a point at a distance from that lens equal to one fourth of its focal length.

The considerations which lead to the forms of the lenses are similar to those mentioned in the case of Huygen's eye piece. The lenses are generally of the forms in the figure, the field glass being plano-convex, the eye glass convexo-plane.

196. The Erecting Eye piece of four lenses is designed to remedy the inverted position of the image in the Astronomical telescope, which renders that instrument if furnished with the eye pieces already described, unfit for viewing terrestrial objects. The manner in which an object is seen through a telescope with this eye piece will, it is expected, be sufficiently understood from the course of a pencil traced in figure 77. The distances, forms, and focal lengths of the lenses are adjusted to diminish as far as possible chromatic and spherical aberration.

197. In the descriptions of the eye pieces they have been supposed to be employed in an Astronomical telescope. The same eye pieces however are employed with the reflecting telescopes. In Galileo's telescope a single eye lens is generally used because the refraction through it is central. The investigations of the field of view will still be true in telescopes with compound eye pieces, if the field glass of the eye piece be used in them instead of the eye lens. The determinations of the magnifying powers will also obtain, if the simple eye lens be supposed such as will refract an excentric pencil in parallel rays at the same inclination to the axis of the lenses as it has at emergence from the eye glass (114).

198. Since the field of view of a telescope is of finite extent, it is necessary to have certain points in the field to which an object observed for the purpose of measurement may be referred. This is in general attained by fixing in the tube of the telescope fine parallel threads in a plane per-

pendicular to the axis of the lenses, which if placed at one of the images formed by the telescope are like that image distinctly visible through the eye glass.

In Huygen's eye piece the wires would be placed at the principal focus of the eye glass, and therefore would be distorted by excentrical refraction through that lens alone; while the image seen would be distorted by excentrical refraction through the field glass and eye glass, and consequently in a different degree from the wires. In this case the position of any point of the field would be incorrectly estimated by referring it to the wires, and thus Huygen's eye piece can never be used in a telescope intended for measuring.

In Ramsden's eye piece the image is formed before the field glass, and at this image the wires are placed. The image and the wires are thus each seen by two excentrical refractions, and are therefore distorted in the same degree, so that the position of a point in the former is correctly estimated by referring it to the latter. This therefore is the eye piece in telescopes used for obtaining measurements.

Galileo's telescope can never be employed in measuring, because the image is a virtual one behind the eye lens.

199. In an Astronomical or Galileo's telescope by moving the eye glass inwards, so as to bring it nearer to the object glass, the pencil from any point of the image emerges from the eye glass in a state of divergence (91), and therefore adapted for a short sighted eye. In viewing a near object, so that a pencil from any point of it cannot at incidence on the object glass be supposed to consist of parallel rays but has a sensible divergency, the eye glass must be moved outwards. The same adjustments are effected in the telescopes of Gregory and Cassegrain by moving the small mirror by a fine screw.

200. To determine practically the magnifying power of a telescope.

If the light of the sky fall upon the object glass or large mirror of a refracting or reflecting telescope, a real image

of that lens or mirror is formed by the eye piece in the same manner as of a self luminous object in the same position. The magnifying power of the telescope is equal to the quotient of the diameter of the object glass or mirror divided by the diameter of its image thus formed. The former diameter can be directly determined, and if the latter be measured by a contrivance called a Dynameter, the numerical magnifying power of the instrument is obtained.

The fact that the magnifying power of a telescope is equal to the quotient of the diameter of its object glass or large mirror divided by the diameter of the bright image of the same, may be separately proved for each telescope with any given eye piece. It may suffice to shew its truth in one case, which shall be that of Gregory's telescope, with a simple eye lens, of which the magnifying power is investigated in (182).

Let pqr (fig. 78) be the image of the large mirror ACB formed by the small mirror whose center is E very nearly at its principal focus, $p'q'r'$ the bright image of pqr which the eye glass whose center is c forms, and which from the largeness of CE may be considered at the principal focus of the eye glass. Then if f_o, f_m, f_e be the focal lengths without reference to algebraic sign of the large mirror, small mirror, and eye glass respectively, by triangles which are very nearly rectilinear and similar,

$$\frac{\text{diameter of mirror}}{pr} = \frac{CE}{Eq},$$

$$\text{and } \frac{pr}{p'r'} = \frac{cq}{cq'}.$$

$$\therefore \frac{\text{diameter of mirror}}{p'r'} = \frac{CE}{Eq} \cdot \frac{cq}{cq'},$$

$$= \frac{f_o + f_m}{f_m} \cdot \frac{f_o}{f_e} \text{ approximately,}$$

$$= \text{magnifying power of telescope (182).}$$

This method is not applicable to Galileo's telescope, because the image of the object glass which the eye lens forms is virtual.

Microscopes.

201. Some objects are so minute that when they are viewed by the naked eye at the least distance of vision, the distances of their parts subtend no appreciable angles, and therefore cannot be discerned. In these cases it is advantageous to view an image of the object instead of the object itself, and an instrument for this purpose is called a microscope.

202. Microscopes are called simple or compound according as a real image of an object viewed by them is not or is formed.

A single lens or sphere forms a simple microscope. If an object be placed nearer to a convex lens than its principal focus, an erect and magnified image of it may be seen by an eye on the axis of the lens (160). Also if a small object PQ (fig. 79) be placed nearer to the center of a refracting sphere than its principal focus, a pencil diverging from a point P will after direct refraction through the sphere diverge very nearly from some point p in CP produced and pq an erect image of PQ is thus formed. This image, if its distance from an eye placed close to the sphere be not less than the least distance of distinct vision, may be seen by the eye by direct pencils, the aperture of the sphere being if necessary limited as in fig. 80 by filling the grooves a, b with an opaque substance, so that the axis of the pencil which reaches the eye may have been refracted through the center of the sphere.

203. A simple microscope preferable to a single lens is composed of two convex lenses separated by a small distance and having a common axis. If an object be placed nearer to the first lens than its principal focus, so that a virtual image of it may be formed by each lens, the image formed by the second lens will be distinctly seen by an eye whose axis is the axis of the lenses, and whose distance from this image is not less than the least distance of distinct vision. This is the principle of Wollaston's Microscopic doublet.

204. The compound refracting microscope is an Astronomical telescope adapted for viewing near objects.

ACB (fig. 81) is a convex lens called the object glass, and acb a convex lens called the eye glass fixed in a tube whose axis is the axis of the lenses. The distance of the centers of the lenses admits of being altered for the purpose of adjustment.

If the axis of the lenses be directed to a point Q in an object PQ which is farther from the object glass than its principal focus, the pencil from a point P after refraction through the object glass converges very nearly to a point p in PC produced, and thus pq an inverted image of PQ is formed. The position of the eye glass is such that this image is at its principal focus, and therefore the pencil from any point p of the image consists after excentrical refraction through the eye glass of rays parallel to pc , and is fitted to render the point p visible to an eye applied to the eye glass. Thus an inverted image of PQ is seen through the microscope.

Obs. A compound eye piece is in general employed for reasons similar to those which render it necessary in a telescope. The forms and focal lengths of the lenses are adjusted on the principles of which the application has been explained with reference to the latter instrument.

205. The Camera Obscura.

If in an aperture in the wall of a darkened room there be inserted a single convex lens or a combination of lenses of considerable negative focal length, a real image of external objects is formed at a distance from the lens. If this image be received on a screen, either directly, or after the direction of the pencils has been altered by reflection at a plane mirror, an inverted picture of external objects is visible.

206. If an object be placed before a convex lens or combination of lenses at a distance a little greater than that of the principal focus, and be illuminated by the sun or a powerful artificial light, a real inverted and magnified image of the object is formed, and if received upon a screen in a darkened room will be seen as a picture upon the screen. This is the principle of the Solar Microscope and the Magic Lantern.

207. The Camera Lucida. Fig. 82.

$ABCD$ is a section of a quadrilateral prism of glass made by a plane perpendicular to the four planes which bound it; the angle A is a right angle, the angle C is 135° , and each of the angles B and D is $67^\circ 30'$. The surface AD , except a small portion near D , is blackened so as not to allow the passage of light.

Let PQ be a luminous object placed before the side AB . The axis of a pencil from a point P of this object after passing nearly perpendicularly through AB is incident on CB at an angle exceeding the critical angle, which between air and glass is about $41^\circ 49'$, and therefore is totally reflected; in a similar manner it is again totally reflected at CD , and then emerges through AD . If pq be a screen parallel to AD , and if a pencil from a point p of it after refraction through the prism near to D emerge in the same direction with the pencil from P , then if the screen be sufficiently distant the image of P and the point p of the screen are seen together by an eye at D , and a representation of the object PQ is visible on the screen.

208. Let radii of a sphere parallel to the faces of the prism and to the normals to its four surfaces in the order according to which the light falls upon them meet the surface of the sphere in I, A, B, C, D (fig. 83). Then I is the pole of the great circle $ABCD$, and $AC = \frac{\pi}{8} = BD$, $BC = \frac{\pi}{4}$.

Also let radii parallel to the axis of a pencil before and after refraction into the prism, before and after refraction out of the prism, meet the sphere in P, Q, S, T , these radii being drawn in the direction opposite to that in which the light proceeds. Let great circles through I and these points meet the circle $ABCD$ in p, q, s, t respectively. Produce DI to D' and join IA, ID' . Draw the great circles $AQP, D'ST$, and also the great circle Ir through the direction of the axis of the pencil after one reflection. Then

$$Bq + Br = \pi, \quad \text{or} \quad Aq + Ar = \frac{\pi}{4};$$

$$Cr + Cs = \pi, \quad \text{or} \quad D's + Ar = \frac{\pi}{4};$$

$$\therefore Aq = D's.$$

$$\text{Also } IQ = IS \text{ (60)}; \quad \therefore QAI = SID';$$

$$\text{and } \frac{\sin TD'}{\sin SD'} = \frac{\sin PA}{\sin QA};$$

\therefore the triangles IPA , ITD' are equal in all respects;

$$\therefore IT = IP, \quad (1)$$

$$\text{and } TID' = PIA;$$

$$\therefore TIP = \frac{\pi}{2}. \quad (2).$$

From (1) it appears that the axis of a pencil refracted and reflected by the prism of a Camera Lucida has at incidence and emergence the same inclination to any edge of the prism, and from (2) it appears that planes parallel to the edges of the prism drawn through the axis of the pencil at incidence and emergence are perpendicular. The effect of the prism therefore is merely to turn through 90° about an axis parallel to an edge of the prism the plane of the axis of any pencil, while in this plane the axis preserves the same direction as before relatively to the same edge. Hence the picture seen on the screen pq is the same as the projection of the object PQ (fig. 82) upon a plane parallel to AB .

209. Hadley's Quadrant or Sextant. Fig. 84.

AC , AB are bars of metal united at A the center of a circular arc CB of from 50° to 70° , to which they are attached at C and B . AD another bar turning about a hinge at A , and carrying a pointer D and vernier along the arc BC . At F and A are two plane reflectors whose surfaces are perpendicular to the plane ABC : the former is fixed to AC , the latter is moveable with AD , and is parallel to F when the pointer D coincides with the point E

of the arc CB . Hence in any other position the angle DAE is the inclination of the mirrors to one another. Of the mirror F the lower part only is silvered, so as to allow the passage of direct rays close to the edge of this reflecting part. G is a small telescope attached to AB , the axis of its lenses being parallel to the plane ABC and passing through the division between the silvered and unsilvered parts of F .

The instrument is used to measure the angular distance between two points.

Let P , Q be two points whose angular distance is required. The plane ABC being brought into the same plane with them, and the telescope pointed to Q , let AD be moved until P seen through the telescope by a pencil reflected in succession at A and F , appears to coincide with Q which is seen directly. In this case the deviation of the axis of the pencil is the angular distance of P and Q . But the deviation of the axis is double the inclination of the mirrors (58), or double the angle DAE . Hence if EC be graduated from E as the zero point, every half degree being marked as a whole one, the reading corresponding to the position of the pointer D will be the angular distance of P and Q .

OBS. The mirrors F and A are called the horizon glass, and the index glass respectively.

COR. If when the pointer D is at E the zero point, the planes of the mirrors, supposed perpendicular to the plane ABC , are not accurately parallel, the angular distance of two objects determined by the instrument will be affected with a constant error called the index error. This correction to observations made by the quadrant is equal to the reading of the limb when the mirrors are exactly parallel, which is the case when they are so adjusted that a very distant bright point seen directly through the telescope, coincides with its image formed by reflection at the two mirrors.

210. The Reflective Goniometer.

The goniometer is an instrument for measuring the angle between two plane faces of a crystal, and consists of a circle of metal turning about an axis perpendicular to its plane. The rim of the circle is graduated, and is read by a pair of verniers in opposite positions. Let the crystal be attached to the circle, so that the plane of the latter is perpendicular to the intersection of the faces of the former whose inclination is required. Bring the circle into such a position that the image of a well defined straight line perpendicular to the plane of the circle formed by reflection at one of the faces of the crystal, coincides with another well defined parallel straight line which is seen directly, and read the verniers. Turn the circle until a similar coincidence is made between the same straight line seen directly and the image of the other formed by reflection at the other face of the crystal, and read the verniers again. The semi-sum of the differences of the two readings of each vernier is the angle through which the circle has been turned, and is equal to the angle between the normals to the two faces of the crystal, and supplemental to the inclination of the two faces.

SECTION VII.

ON THE RAINBOW.

211. IF a pencil of light be refracted into a sphere, when it is incident on the interior surface of the sphere a portion of it emerges and another portion is internally reflected; this latter portion being again incident on the interior surface is partially reflected and partially refracted, and so on continually. The intensity of light in the pencils which thus successively emerge rapidly decreases.

212. Let C (fig. 95) be the center of a sphere of water, the refractive index of which out of air for rays of mean refrangibility is 1.335 or $\frac{4}{3}$ very nearly. Let a pencil of parallel rays of homogeneous light whose axis is in direction AC be incident directly on the sphere. Take

$$Cq_1 = 3AC, \quad Cq_2 = \frac{3}{5}AC, \quad Cq_3 = AC; \quad (31, 33)$$

then q_1, q_2, q_3 are the geometrical foci of the pencil when refracted into the sphere, reflected at the internal surface, and emergent from the sphere respectively. Neglecting aberration we may consider the pencil of mean rays which emerges after one internal reflection as diverging from q_3 .

If this divergent pencil fall upon an eye E , (fig. 97) its pupil will select a small portion whose axis has been passed in direction $PQRSE$. If PQ the axis of this small pencil be produced to any point V , and ES produced to meet PV in T , ETV is the deviation of the axis of the pencil.

213. Let ϕ , ϕ' , be the angles of incidence and refraction of the axis PQ of the small pencil which is refracted to the eye after one internal reflection, D the deviation of the axis. The deviation at each of the two refractions being $\phi - \phi'$, and the deviation at the reflection $\pi - 2\phi'$,

$$\therefore D = 2(\phi - \phi') + \pi - 2\phi'.$$

$$\therefore d_{\phi}D = 2 - 4 \frac{\cos \phi}{\mu \cos \phi'},$$

$$d_{\phi}^2 D = \frac{4 \sin \phi}{\mu \cos \phi'} \left\{ 1 - \left(\frac{\cos \phi}{\mu \cos \phi'} \right)^2 \right\}.$$

If D be a minimum or maximum,

$$\begin{aligned} 4 \cos^2 \phi &= \mu^2 \cos^2 \phi' \\ &= \mu^2 - \sin^2 \phi, \end{aligned}$$

$$\therefore 4 - 3 \sin^2 \phi = \mu^2,$$

$$\sin \phi = \sqrt{\frac{4 - \mu^2}{3}}.$$

Now $\cos \phi'$ is $> \cos \phi$ and $\therefore > \frac{\cos \phi}{\mu}$,

$\therefore d_{\phi}^2 D$ is positive with this value of ϕ ,

or D is a minimum.

In this case since $d_{\phi}D = 0$, the variation of deviation for a small variation of ϕ is extremely small, or the emergent pencil which reaches the eye may be considered to consist as at incidence on the sphere of parallel rays. In no position of the drop, as has been shewn (212), can the pencil be convergent; in other positions therefore than that which renders D a minimum the pencil which enters the eye is divergent.

From the value of ϕ which has been obtained it appears that R is the primary focus after refraction into the sphere of the pencil which emerges in parallel rays. (51).

214. Again let C (fig. 96) be the center of a sphere of water, into which a pencil of mean rays parallel to AC is directly refracted. Take

$$Cq_1 = 3AC, \quad Cq_2 = \frac{3}{5}AC, \quad Cq_3 = \frac{3}{11}AC, \quad Cq_4 = \frac{2}{5}AC;$$

the latter two lines being measured in a direction contrary to that of the incident pencil.

Then q_1, q_2, q_3, q_4 are the geometrical foci of the pencil when first refracted, once reflected, twice reflected, and emergent after two internal reflections respectively. Neglecting aberration we may consider the pencil which emerges after two internal reflections as diverging from q_4 .

If this divergent pencil fall upon an eye E , (fig. 98) its pupil selects a small portion whose axis has passed in direction $PQRSTE$. If PQ the axis of this small pencil be produced to any point V , and ET cut QV in T , the outward angle ETV or $\pi + PTE$ is the deviation of its axis.

215. If ϕ, ϕ' , be the angles of incidence and refraction of PQ the axis of the small pencil which is refracted to the eye after two internal reflections, D the deviation of its axis,

$$D = 2(\phi - \phi') + 2(\pi - 2\phi'),$$

$$d_\phi D = 2 - 6 \frac{\cos \phi}{\mu \cos \phi'},$$

$$d_\phi^2 D = \frac{6 \sin \phi}{\mu \cos \phi'} \left\{ 1 - \left(\frac{\cos \phi}{\mu \cos \phi'} \right)^2 \right\}.$$

Therefore if D be a maximum or minimum,

$$9 \cos^2 \phi = \mu^2 \cos^2 \phi' = \mu^2 - \sin^2 \phi,$$

$$\sin \phi = \sqrt{\frac{9 - \mu^2}{8}},$$

which renders $d_\phi^2 D$ positive, and therefore D a minimum. In this case the pencil which enters the eye may be considered to consist of parallel rays; in other cases it is divergent.

From the value of ϕ which has been obtained, it appears that the pencil which finally emerges in parallel rays has, after refraction into the sphere, a primary focus at a distance from $Q = \frac{3}{4} QR$, that between the two reflections it consists of parallel rays, and that after the second reflection it has a primary focus at a distance from $S = \frac{3}{4} QR$. (48, 51).

216. The propositions just investigated enable us to explain the formation of a Rainbow.

Let A (fig. 97) be the center of a small spherical drop of rain falling in the air, and let a pencil of sun light fall upon it, which from the distance of the sun regarded as a point, may be considered to consist of parallel rays. Suppose light homogeneous. Then of this pencil a small pencil with PQ for its axis after being refracted into the sphere and once internally reflected, may emerge in a divergent state in the direction TE (212) and fill the pupil of an eye E , creating the sensation of a bright point in the sky in direction ET of the colour belonging to the species of light which is considered. Through E draw EB parallel to PQ ; then since the deviation of PQ , and consequently the angle TEB , depend merely on the angle of incidence of PQ , and since all rays from the sun may be considered parallel, therefore all drops of rain whose centers lie in a conical surface with EB for its axis and TEB for its semi-vertical angle, will transmit to the eye a similar pencil of divergent rays. A drop of rain at less angular distance from EB than A will transmit to the eye a small pencil with greater deviation and consequently greater divergence than TE , and therefore producing the sensation of a less bright point in the sky, and the divergence of pencils which reach the eye at a greater angular distance from EB than A is less than that of the pencil TE , until when the deviation is a minimum, the pencil is nearly one of parallel rays, and therefore produces the greatest impression upon the eye.

The appearance therefore to the eye would be an illuminated sky of the colour of the light which has been considered, the brightness being greater at greater distances from the line EB until it is bounded by a circle whose center is in EB ,

beyond which there is no colour and comparative darkness. For each species of sun light this would be the appearance, the bounding circles for different species differing slightly in position, since the amount of minimum deviation depends on the refractive index of the light (213). The result of superposing these illuminations of different colours will be white light within a certain distance from the line EB , terminated by a narrow circular band of vivid colours arranged in concentric circles about the line EB . Beyond this band is comparative blackness. This phenomenon produced by pencils which have been once internally reflected in the raindrops is called the Primary Rainbow.

217. In a similar manner it may be shewn that pencils twice internally reflected in the falling raindrops may enter the eye in a state of divergence (214). The deviation of the axis $PQRTE$ of such a pencil is the outward angle QTE (fig. 98), which is greater and therefore the divergence of the pencil is greater at greater angular distances from EB . The appearance presented to the eye will therefore be white light beyond a certain distance from EB terminated by a narrow circular band of vivid colours, within which there is comparative darkness. This phenomenon is called the Secondary Rainbow.

218. The theoretical existence of rainbows caused by pencils which have been three or more times internally reflected may be similarly shewn, but the rapid decrease in the intensity of light in the emergent pencils renders such rainbows with very few exceptions invisible.

In the explanation of the formation of a rainbow the sun has been considered a point. To every point in the sun's disc which is of finite extent there will be a corresponding rainbow, and the visible rainbow resulting from the superposition of these will have its colours in some degree confused, but its general appearance such as has been described.

219. DEF. In any rainbow the semi-vertical angle of a cone of raindrops which transmits to the eye pencils of

parallel rays of a given colour, is the radius of the bow for that colour.

220. DEF. In any rainbow the angular elevation above the horizon of the highest raindrop which transmits to the eye a pencil of parallel rays of a given colour is the altitude of the bow for that colour.

Hence the altitude of any colour in a rainbow added to the altitude of the sun's center is equal to the radius of the bow for the same colour. If the altitude of the sun exceed the radius of the bow, the horizon will render the rainbow invisible.

If D_1 be the minimum deviation of light of a given kind after one internal reflection, α the altitude of the corresponding colour in the primary bow, then

$$D_1 + \alpha = 180^\circ - \text{the altitude of the sun's center.}$$

If D_1 be the corresponding deviation in the secondary bow for the colour whose altitude therein is α ,

$$D_1 - \alpha = 180^\circ + \text{the altitude of the sun's center.}$$

221. To investigate the order of the colours in the primary and secondary rainbow.

Let the axis of a pencil of sun light whose refractive index is μ be incident on a raindrop at an angle ϕ , and emerge after p internal reflections. Let ϕ' be the angle of refraction corresponding to the angle of incidence ϕ , and D the deviation of the axis of the pencil. Since $\phi - \phi'$ is the deviation produced in the axis of the pencil by each of the two refractions, and $\pi - 2\phi'$ that produced by each of the p reflections,

$$\therefore D = 2(\phi - \phi') + p(\pi - 2\phi'), \text{ and } \sin \phi = \mu \sin \phi'.$$

If the pencil consists at emergence of parallel rays, $d_\phi D = 0$,

$$\therefore 0 = 1 - (p + 1) d_\phi \phi';$$

$$0 = \cos \phi - \mu \cos \phi' d_\phi \phi',$$

$$\therefore 0 = \mu \cos \phi' - (p + 1) \cos \phi \quad (1).$$

Let D_1 be the deviation in this case corresponding to the extremity of the colour considered. In examining its alteration produced by an alteration of μ , ϕ and ϕ' must be regarded in consequence of the latter circumstance as functions of μ .

$$\therefore d_\mu D_1 = 2 \{d_\mu \phi - (p + 1) d_\mu \phi'\}.$$

$$\text{But } \cos \phi d_\mu \phi = \sin \phi' + \mu \cos \phi' d_\mu \phi';$$

$$\begin{aligned} \therefore d_\mu D_1 &= \frac{2}{\cos \phi} \{ \sin \phi' + \mu \cos \phi' d_\mu \phi' - (p + 1) \cos \phi d_\mu \phi' \}. \\ &= \frac{2 \sin \phi'}{\cos \phi} \text{ from (1),} \\ &= \frac{2}{\mu} \tan \phi, \end{aligned}$$

therefore $d_\mu D_1$ is positive or the deviation of the axis of the pencil is greater as μ is greater.

Hence in the primary rainbow, since the minimum deviation is least for red and greatest for violet light, the red circle is highest and the violet lowest (220). In the secondary bow the red circle, for which the deviation is least, is lowest, and the violet circle highest.

222. The investigations of this chapter are to be received as general explanations of the rainbow rather than as exact calculations of its phenomena. The divergence of the oblique pencil emergent from the raindrop has been inferred from the position of the geometrical focus of the direct pencil, and the possibility of the emergent light having maximum intensity at other places besides the edge of the bow has not been considered.

A pencil of parallel rays incident on a raindrop and emergent after one or more internal reflections forms a caustic surface, and the small oblique part of it which enters the eye is determined by drawing from the eye a tangent to the caustic. To find therefore the illumination at different points of the sky it is necessary to calculate the intensity of light at dif-

ferent points of the caustic on the principles of physical optics. (Cambridge Philosophical Transactions, Vol. 6). It is thus found that in the case of the primary bow the principal maximum of illumination lies a little within the position which the geometrical theory gives; also that within this there is a series of inferior maxima becoming in succession smaller. Hence when compound light is considered there will be in addition to the principal rainbow a series of interior bows decreasing in brilliancy. To these the name of spurious or supernumerary bows is given.

Two or three of the spurious bows of the primary rainbow may sometimes be seen in the case of nature. The results of theory in this subject are tested by measurement by allowing a very thin column of water to run from a vessel, and receiving by a telescope artificial light which has been internally reflected in this column. (Cambridge Phil. Trans. Vol. 7).

APPENDIX.

223. To find the velocity of light from astronomical elements.

Let α = the coefficient of aberration in seconds,

β = the sun's mean equatoreal horizontal parallax...,

R = radius of the earth's equator in miles,

P = periodic time of the earth in seconds of time,

$$= 31558149.60768.$$

$$\begin{aligned} \text{Now } \frac{\text{mean vel. of earth}}{\text{vel. of light}} &= \text{circular measure of } \alpha, \\ & \quad (\text{Hymers' Ast. 286.}) \\ &= \alpha \sin 1'', \end{aligned}$$

$$\text{mean distance of the earth from the sun} = \frac{R}{\beta \sin 1''} \quad (\text{Hymers 246}),$$

$$\therefore \text{mean velocity of the earth} = \frac{2\pi R}{P\beta \sin 1''} \text{ miles per second,}$$

$$\text{and velocity of light} = \frac{2\pi R}{P\beta \alpha \sin^2 1''} \text{ miles per second.}$$

The numerical value of β is $8.5776''$, and $2R = 7924$ miles.

If $\alpha = 20.35''$ the value of the constant of aberration adopted by the Astronomical Society,

$$\text{velocity of light} = 192222 \text{ miles per second.}$$

If $\alpha = 20.5''$, the constant of aberration obtained by Mr Richardson,

$$\text{velocity of light} = 190859 \text{ miles per second.}$$

224. It is found by experiment that luminous surfaces appear equally bright at any point, whatever be the inclination of the surface at that point to the axis of the pencil by which it is seen. The sun's disc for example is equally bright at all distances from the center. Hence it must follow that the copiousness of emission of light from a luminous surface must be proportional to the sine of the inclination between the direction of emission and the surface.

225. Let A (fig. 2) be a small plane area illuminated by a surface BC of uniform brightness. About A as center describe a sphere, and let a line through A moving round the boundary of BC , intersect the surface of this sphere in the curve bc . Take also an element P of the surface BC and let p be the corresponding element of the spherical surface formed as before. Let α , θ be the inclinations of AP to the illuminated plane, and to the surface at P .

If the element at P be regarded as an origin of light,

$$\text{illumination at } A \text{ from it} = C \frac{\sin \alpha}{AP^2} \times \text{area } P \sin \theta; \quad (14, 224)$$

and if the surface of the sphere be supposed of the same uniform brightness with BC and the element at p be considered an origin of light,

$$\text{illumination at } A \text{ from it} = C \frac{\sin \alpha}{Ap^2} \text{area } p.$$

$$\text{But } \frac{\text{area } P \cdot \sin \theta}{AP^2} = \frac{\text{area } p}{Ap^2};$$

or the illuminations at A from corresponding elements of BC and bc are equal; therefore the illumination from BC is the same as that from bc .

But the surface of the sphere at p being inclined to that of A at an angle $\frac{\pi}{2} - \alpha$,

$$\text{area } p \times \sin \alpha = \text{area of projection of } p \text{ on plane of } A;$$

\therefore illumination at A from p

$$= \frac{C}{Ap^2} \times \text{area of projection of } p \text{ on plane of } A;$$

\therefore illumination at A from BC

$$= \text{illumination at } A \text{ from } bc$$

$$= \frac{C}{Ap^2} \times \text{area of projection of } bc \text{ on plane of } A.$$

If therefore the area of the projection of bc can be found, the illumination at A is known.

Ex. If A be illuminated by an equally bright sky in all directions above the horizon,

$$\text{area of projection of } bc = \pi \cdot Ap^2;$$

$$\therefore \text{illumination at } A = \pi C.$$

226. A plane surface touches a self-luminous sphere; to find the illumination of the surface at any point.

Let A (fig. 99) be any point in the plane surface which the sphere whose center is C touches at P . Join CP . With center A describe a spherical surface and let a straight line through A moving round the sphere C so as to define the portion of it from which A receives illumination, intersect the spherical surface about A in the small circle pq , the plane of which meets AC or AC produced in c .

Then projection on the plane of curved surface pq

$$= \text{projection of the plane circle } pq$$

$$= \pi \cdot cp^2 \cdot \frac{cp}{Ap} \cdot (\text{Hymers' Anal. Geom. 81}).$$

∴ illumination at A from the sphere

$$\begin{aligned}
 &= \frac{C}{Ap^2} \cdot \frac{\pi \cdot cp^3}{Ap} \quad (225) \\
 &= C \cdot \left(\frac{CP}{AC} \right)^3.
 \end{aligned}$$

227. When a pencil is directly reflected at a spherical surface, to find the limits of the position of the origin that the pencil may converge after reflection.

With the notation of (26) the position of the geometrical focus of the reflected pencil is given by

$$\frac{1}{v} = \frac{2}{r} - \frac{1}{u}.$$

Hence in order to v being positive, or the reflected pencil convergent,

(1) if r be positive, or the mirror concave, u must be either negative, or positive and greater than $\frac{r}{2}$, i.e. the origin must not lie between the surface and the principal focus;

(2) if r be negative, or the mirror convex, u must be negative and numerically less than $\frac{r}{2}$, i.e. the origin must lie between the surface and the principal focus.

OBS. From these results may be seen the reason of the position of the mirrors in the telescopes of Gregory and Cassegrain.

228. If a small pencil of diverging rays be reflected at a concave spherical mirror, to find the limits of the distance of the origin from the point of incidence in order that the reflected pencil may converge to or diverge from both the focal lines.

The distances of the primary and secondary foci of the reflected pencil from the point of incidence of its axis are given by the equations

$$\left. \begin{aligned} \frac{1}{v_1} &= \frac{2}{r \cos \phi} - \frac{1}{u} \\ \frac{1}{v_2} &= \frac{2 \cos \phi}{r} - \frac{1}{u} \end{aligned} \right\}. \quad (48).$$

Hence the reflected pencil converges to each focal line, or v_1 and v_2 are both positive, if u be greater than $\frac{r}{2 \cos \phi}$ and consequently greater than $\frac{r \cos \phi}{2}$; and it diverges from each line, or v_1 and v_2 are both negative, if u be less than $\frac{r \cos \phi}{2}$ and so less than $\frac{r}{2 \cos \phi}$. If the distance of the origin be greater than $\frac{r \cos \phi}{2}$ and less than $\frac{r}{2 \cos \phi}$, the reflected pencil converges to one of the lines and diverges from the other.

If the rays reflected in the primary plane are parallel, or v_1 infinite,

$$\begin{aligned} \frac{1}{u} &= \frac{2}{r \cos \phi}, \\ \text{and } \frac{1}{v_2} &= -\frac{2 \sin^2 \phi}{r \cos \phi}, \end{aligned}$$

which is the polar equation to the locus of the secondary focus, when the origin assumes different positions in the same primary plane so that the rays reflected in the primary plane may be parallel.

229. A pencil passes directly from air into water through a plate of glass, to find its geometrical focus, supposing the indices of refraction from air and water into glass to be $\frac{3}{2}$ and $\frac{4}{3}$ respectively.

Let Q (fig. 34) be the origin of the pencil whose axis cuts the surfaces of the glass plate perpendicularly in A, B ; F_1 the geometrical focus of the pencil after refraction into the plate, F its geometrical focus at emergence into water.

Then $AF_1 = \frac{3}{2} AQ, \quad (68)$

and since $\frac{9}{8}$ is the index of refraction from water into glass (67),

$$BF_1 = \frac{9}{8} BF;$$

$$\therefore BF = \frac{8}{9} AB + \frac{4}{3} AQ.$$

230. A pencil of parallel rays is directly refracted through an equiconvex lens of 20 inches focal length and 4 inches aperture, the refractive index of the substance of which it is formed being 1.5, to find the aberration of the extreme rays.

By (92) the aberration required,

$$= -\frac{\mu - 1}{\mu^2} \left\{ \frac{1}{r^3} - \left(\frac{1}{s} - \frac{1}{f} \right)^2 \left(\frac{1}{s} - \frac{\mu + 1}{f} \right) \right\} \frac{f^2 y^2}{2}.$$

Here $\mu = 1.5, \quad f = -20, \quad y = 2,$

$$\frac{2}{f} = \frac{1}{r} - \frac{1}{s} \quad \text{or} \quad -\frac{1}{10} = \frac{2}{r}; \quad \therefore r = -20, \quad s = 20;$$

$$\therefore \text{aberration} = \frac{1}{9} \text{ inch.}$$

231. A direct pencil of rays converges to a point at the extreme distance from a lens at which an absence of aberration in a converging pencil is possible; to determine the relation between the radii of the lens that there may be no aberration in the refracted pencil, the index of refraction being 1.5.

If the pencil be refracted without aberration, then with the notation of (93)

$$\mu(\mu + 2)x^2 - 4\mu(\mu^2 - 1)ax + (3\mu + 2)\mu(\mu - 1)^2a^2 + \mu^4 = 0.$$

$$\therefore x = 2 \frac{\mu^2 - 1}{\mu + 2} a \pm \sqrt{4 \left(\frac{\mu^2 - 1}{\mu + 2} \right)^2 a^2 - \frac{\mu^3}{\mu + 2} - \frac{(3\mu + 2)(\mu - 1)^2}{\mu + 2}}.$$

If a have a value which limits the possibility of aberration being destroyed,

$$4 \left(\frac{\mu^2 - 1}{\mu + 2} \right)^2 \alpha^2 = \frac{\mu^3 + (3\mu + 2)(\mu - 1)^2}{\mu + 2},$$

$$\therefore \alpha = \pm 1.67, \mu \text{ being } 1.5.$$

The pencil being convergent at incidence, $\frac{1}{u} = \frac{\alpha - 1}{2f}$ is negative. Since then α is numerically greater than unity, its negative value must be used;

$$\therefore x = 2 \frac{\mu^2 - 1}{\mu + 2} \alpha = -1.19.$$

$$\therefore \frac{r}{s} = \frac{x - 1}{x + 1} = \frac{219}{19},$$

the relation between the radii of the lens.

232. A luminous point lies in the axis produced of a cone of glass, the vertical angle of which is 90° ; to find the caustic surface produced by the refracted rays.

Let Q (fig. 100*) be the luminous point, A the vertex of the cone, QR a ray incident on the cone at R and refracted in direction qR , ϕ, ϕ' its angles of incidence and refraction. Draw RM perpendicular to the axis of the cone, and let $AM = x = RM$, $AQ = b$.

If X, Y be the co-ordinates of any point in qR ,

$$\begin{aligned} \therefore \frac{Y - y}{X - x} &= \tan RqA = -\cot \left(\frac{\pi}{4} + \phi' \right) \\ &= \frac{\sin \phi' - \cos \phi'}{\sin \phi' + \cos \phi'}. \end{aligned}$$

$$\left(\frac{X - Y}{2x - X - Y} \right)^2 = \left(\frac{\cos \phi'}{\sin \phi'} \right)^2 = \frac{1}{\sin^2 \phi'} - 1,$$

$$\left(\frac{X - Y}{2x - X - Y} \right)^2 + 1 = \frac{1}{\sin^2 \phi'}.$$

$$\text{Also } \frac{x}{x+b} = \tan RQA = \frac{\sin \phi - \cos \phi}{\sin \phi + \cos \phi},$$

$$\therefore \left(\frac{b}{2x+b} \right)^2 + 1 = \frac{1}{\sin^2 \phi}.$$

$$\therefore \mu^2 \left(\frac{b}{2x+b} \right)^2 + \mu^2 = \left(\frac{X-Y}{2x-X-Y} \right)^2 + 1 \quad (1)$$

is the equation to qR wherein x is the parameter. If we differentiate with regard to this parameter, the equation

$$\mu^2 \frac{b^2}{(2x+b)^3} = \frac{(X-Y)^2}{(2x-X-Y)^3} \quad (2)$$

in conjunction with the former gives the relation between X and Y when they are the co-ordinates of the point of intersection of qR with the consecutive refracted ray in the plane QAR .

From equation (2)

$$\frac{b}{2x+b} = \frac{b^{\frac{1}{3}} \mu^{\frac{2}{3}} b^{\frac{2}{3}} - (X-Y)^{\frac{2}{3}}}{\mu^{\frac{2}{3}} X + Y + b},$$

$$\frac{X-Y}{2x-X-Y} = (X-Y)^{\frac{1}{3}} \cdot \frac{\mu^{\frac{2}{3}} b^{\frac{2}{3}} - (X-Y)^{\frac{2}{3}}}{X + Y + b}.$$

Therefore after substitution in (1)

$$\mu^2 - 1 = \frac{\{(X-Y)^{\frac{2}{3}} - \mu^{\frac{2}{3}} b^{\frac{2}{3}}\}^3}{(X+Y+b)^2},$$

a relation between X and Y independent of any particular position of the point R , and therefore the equation to the curve by the revolution of which about AM the caustic surface is generated.

233. Rays issuing from a luminous point in its axis are incident on a thin lens. A portion of those that enter the lens is allowed to proceed at once through the second surface; a second portion however does not escape until it

has been twice internally reflected; a third portion four times reflected; a fourth portion six times, and so on. To shew that a row of images will be formed whose distances from the lens are in harmonic progression.

The pencil being supposed small, each geometrical focus may be regarded as a new origin. Let A (fig. 100) be the point where the axis of the lens meets its surface, q the geometrical focus which is the origin of the pencil incident on the second surface of the lens the n^{th} time, Q_n the n^{th} image or geometrical focus of the emergent portion, q_1, q_2 the successive geometrical foci of the reflected portion, Q_{n+1} the $n + 1^{\text{th}}$ image, r, s the radii of the surface and μ the index of refraction.

$$\frac{\mu}{Aq} - \frac{1}{AQ_n} = \frac{\mu - 1}{s} \quad (61, 32) \quad (1)$$

$$\frac{1}{Aq_1} + \frac{1}{Aq} = \frac{2}{s} \quad (26) \quad (2)$$

$$\frac{1}{Aq_2} + \frac{1}{Aq_2} = \frac{2}{r} \quad (3)$$

$$\frac{\mu}{Aq_2} - \frac{1}{AQ_{n+1}} = \frac{\mu - 1}{s} \quad (4)$$

From (2) and (3),

$$\frac{1}{Aq_2} - \frac{1}{Aq} = \frac{2}{r} - \frac{2}{s}.$$

From (1) and (4)

$$\begin{aligned} \frac{1}{AQ_{n+1}} - \frac{1}{AQ_n} &= \mu \left(\frac{1}{Aq_2} - \frac{1}{Aq} \right) \\ &= 2\mu \left(\frac{1}{r} - \frac{1}{s} \right) \quad \text{a constant quantity,} \end{aligned}$$

and Q_n, Q_{n+1} are any two successive images; therefore the distances of the images from A are in harmonical progression.

234. An image of a very distant object is formed by a plano-convex lens, the pencils before incidence on the lens passing through a small diaphragm whose center is in the axis of the lens; to find the position of the diaphragm that the image may be distinct.

Let ABO (fig. 101) be the axis of the lens, O the center of its curved surface, B the center of the diaphragm, PBR the axis of a pencil from a point of the object incident upon the lens, the rays of this pencil in consequence of the distance of the object being considered parallel. Indistinctness in the image will arise solely from oblique refraction at the second surface (19), and will be destroyed if the refraction at that surface be direct, or if OR be the direction of the axis of the pencil PBR after refraction at the first surface. In this case if μ be the refractive index of the lens,

$$\mu = \frac{\sin RBA}{\sin ROA} = \frac{AO}{AB}, \quad \text{or} \quad AB = \frac{AO}{\mu},$$

which assigns the position of the diaphragm.

235. A stop is placed on the axis of a concave spherical reflector at a distance from it equal to four times its focal length: to measure the distortion at any point of the image of a very distant object to which the axis is directed.

Let R (fig. 55) be the point of incidence at an angle ϕ of the pencil from a given point of the object whose image is q . Draw RN perpendicular to AX . Then the distances from R of the primary and secondary foci for this point are

$$\frac{r}{2} \cos \phi, \quad \text{and} \quad \frac{r}{2} \sec \phi; \quad (48)$$

$$\therefore Rq = \frac{r}{4} (\cos \phi + \sec \phi) \quad (55) = \frac{r}{2},$$

if powers of ϕ above the second be neglected.

$$\text{Also } AX = 2r,$$

$$\therefore AY = \frac{2r}{3} \left(1 - \frac{y^2}{6r^2} \right); \quad (107)$$

$$\therefore Nm = Rq \cos \frac{y}{AY} = \frac{r}{2} \left(1 - \frac{9y^2}{8r^2} \right);$$

$$\therefore Am = AN + Nm = \frac{r}{2} - \frac{y^2}{16r}.$$

$$\therefore Ym = \frac{r}{6} \left(1 - \frac{7y^2}{24r^2} \right).$$

Also, since $\beta = -\frac{1}{2}$, (107)

$$\frac{\tan \eta}{\tan \epsilon} = 3 \left(1 + \frac{2y^2}{3r^2} \right);$$

$$\begin{aligned} \therefore qm &= Ym \cdot \tan \eta \\ &= \frac{r}{2} \left(1 + \frac{3y^2}{8r^2} \right) \tan \epsilon. \end{aligned}$$

Now the part of the object whereof qm is the image has a size proportional to $\tan \epsilon$; hence the distortion at any point of the image is given by the coefficient of y^2 in the expression

$$\frac{r}{2} \left(1 + \frac{3y^2}{8r^2} \right).$$

236. A pencil of parallel rays is refracted directly through an oblate spheroid, its axis passing perpendicularly to the axis of the spheroid, to find the geometrical focus of any section of the emergent pencil made by a plane through its axis.

Let a, b be the semiaxes of the generating ellipse. At the point of incidence of the axis of the pencil the radii of curvature of the principal normal sections are a and $\frac{b^2}{a}$. If then r be the radius of curvature of a normal section inclined at an angle θ to the equator of the spheroid,

$$\frac{1}{r} = \frac{\cos^2 \theta}{a} + \frac{a \sin^2 \theta}{b^2}.$$

Let v_1, v_2 be the distances from the first and second surfaces of the geometrical foci of this section of the pencil

after refraction at the first surface and at emergence respectively.

$$\text{Then } \frac{\mu}{v_1} = -\frac{\mu-1}{r}, \quad (34)$$

$$\frac{\mu}{v_1 + 2a} - \frac{1}{v_2} = \frac{\mu-1}{r},$$

$$\therefore \frac{1}{v_2} = \frac{\mu-1}{r} \left\{ \frac{\mu}{2(\mu-1)\frac{a}{r} - \mu} - 1 \right\}$$

$$= (\mu-1)a \left(\frac{\cos^2 \theta}{a^2} + \frac{\sin^2 \theta}{b^2} \right) \left\{ \frac{1}{2\frac{\mu-1}{\mu}a^2 \left(\frac{\cos^2 \theta}{a^2} + \frac{\sin^2 \theta}{b^2} \right)} - 1 \right\},$$

which determines the geometrical focus of the section of the pencil defined by the value of θ .

237. A hollow sphere of glass, the radii of whose surfaces are given, is filled with water; to find the geometrical focus of a pencil of parallel rays directly refracted through it.

Let q_1, q_2, q_3 , be the distances from the center of the sphere of the geometrical foci of the pencil after refraction at the first surface of the sphere, after refraction through the sphere of water, and at emergence respectively, these distances being accounted positive when measured in a direction contrary to that of the incident light. Let r, s be the radii of the exterior and interior surfaces of the glass, μ, μ' the indices of refraction from air into glass and water respectively.

$$\text{Then } \frac{1}{q_1} = -\frac{\mu-1}{r}. \quad (33).$$

Again $\frac{\mu'}{\mu}$ being the index of refraction out of glass into water (67),

$$\frac{1}{q_2} - \frac{1}{q_1} = -2\frac{\frac{\mu'}{\mu} - 1}{\frac{\mu'}{\mu}s} \quad (100).$$

Also
$$\frac{\mu}{q_3} - \frac{1}{q_2} = -\frac{\mu - 1}{r} \quad (61, 33).$$

Therefore by addition

$$\begin{aligned} \frac{\mu}{q_3} &= -2 \frac{\mu - 1}{r} - 2 \frac{\mu' - \mu}{\mu' s}, \\ \frac{1}{q_3} &= -2 \frac{\mu - 1}{\mu r} - 2 \frac{\mu' - \mu}{\mu \mu' s}, \end{aligned}$$

which defines the geometrical focus of the emergent pencil.

238. A luminous point of white light being placed on the axis of a lens, to shew that as the point moves along the axis the distance between the geometrical foci for two given colours will pass through a minimum, and to determine the position of the point in this case.

Let u be the distance of the point from the lens, v_1, v_2 the distances of the geometrical foci for the given colours, f_1, f_2 the focal lengths of the lens of the given colours.

$$\text{Then } \left. \begin{aligned} \frac{1}{v_1} &= \frac{1}{u} + \frac{1}{f_1} \\ \frac{1}{v_2} &= \frac{1}{u} + \frac{1}{f_2} \end{aligned} \right\} \begin{matrix} (87) \\ (1). \end{matrix}$$

If $u = -f_1$ or $-f_2$, of the distances v_1 and v_2 one is infinite and the other finite. In either position of the luminous point the distance between the geometrical foci is infinite, and therefore for some position intermediate to these, the distance between the foci is a minimum.

Of the two quantities f_1 and f_2 let f_1 be the greater; v_1 therefore is greater than v_2 .

$$\text{Now } \frac{d_u v_1}{v_1^2} = \frac{1}{u^2}, \quad \frac{d_u v_2}{v_2^2} = \frac{1}{u^2}.$$

$$\therefore d_u (v_1 - v_2) = \frac{1}{u^2} (v_1^2 - v_2^2),$$

$$\text{and } d_u^2 (v_1 - v_2) = \frac{2}{u^4} (v_1^3 - v_2^3) - \frac{2}{u^3} (v_1^2 - v_2^2).$$

If $v_1 - v_2$ be a maximum or minimum,

$$d_u(v_1 - v_2) = 0;$$

$$\therefore v_1^2 = v_2^2,$$

$$\therefore v_1 = -v_2,$$

the solution $v_1 = v_2$ being inconsistent with equations (1).

If v_1 be positive, and $v_2 = -v_1$, $d_u^2(v_1 - v_2)$ is positive and $v_1 - v_2$ is a minimum. In this case equations (1) become

$$\frac{1}{v_1} = \frac{1}{u} + \frac{1}{f_1},$$

$$-\frac{1}{v_1} = \frac{1}{u} + \frac{1}{f_2}.$$

$$\therefore \frac{1}{u} = -\frac{1}{2} \left(\frac{1}{f_1} + \frac{1}{f_2} \right),$$

which assigns the position of the luminous point when the distance between the two foci is the least possible.

239. When a spectrum is measured in Fraunhofer's manner to find the angle subtended through the telescope between the line A in its axis and the line B , having given $\mu_A = \sqrt{3}$, $\mu_B - \mu_A = .001$, the prism having an angle of 60° and the distance of its vertex from the slit and the object glass being 3 and 10 times the focal length of the object glass, the telescope magnifying 20 times.

With the notation of (139) the angle which the lines A and B subtend at the prism

$$= \frac{\sin i}{\cos \phi' \cos \psi} \delta \mu \quad \text{in circular measure,}$$

and since in the present case when the prism has the position of minimum deviation, $\phi' = \psi' = 30^\circ$,

$$\text{and } \cos \psi = \sqrt{1 - 3 \sin^2 30^\circ} = \frac{1}{2},$$

therefore the angle between A and $B = .002$.

If then f be the focal length of the object glass, the linear distance of A and B in the image formed the prism is $3f \times .002$ or $.006.f$. This distance subtends at the object glass of the telescope the angle $\frac{.006}{13}$, and consequently the corresponding part of its image formed by the telescope and viewed by the eye subtends at the eye the angle whose circular measure is $\frac{.006}{13} \times 20$ or $.0923$.

240. A small plane reflector stands upon a horizontal plane and is inclined at a given angle to it; to determine how great a length of his person a man standing before it at a given distance from it can see.

Let $AB = a$ be the height of the man (fig. 102), BD the horizontal plane on which he stands, DE the mirror inclined at a given angle a to the vertical, $BD = b$, $DE = c$. Let ab be the image of the man constructed under the condition of each point of it being at the same perpendicular distance from DE or DE produced as the corresponding point of AB (20). If AD , AE be produced to intersect this image in F and G , FG is the length of his person which the man sees. Produce AB , ED , ab to meet in C ; join Aa intersecting DE produced in H , and draw FK parallel to DE to meet AG in K .

$$\text{Then } ACD = a, \quad AC = a + b \cot a,$$

$$\therefore AH = AC \sin a = a \sin a + b \cos a,$$

$$\text{and } CH = AC \cos a = (a + b \cot a) \cos a,$$

$$\therefore EH = CH - CD - ED = (a + b \cot a) \cos a - b \operatorname{cosec} a - c.$$

$$\begin{aligned} \frac{FG}{FK} &= \frac{\sin AEH}{\sin (AEH + a)} \\ &= \frac{1}{\sin a \cot AEH + \cos a} \\ &= \frac{AH}{EH \sin a + AH \cos a} \\ &= \frac{a \sin a + b \cos a}{a \sin 2a + b \cos 2a - c \sin a} \end{aligned}$$

$$\frac{FK}{DE} = \frac{AF}{AD} = \frac{AC}{AD} \cdot \frac{\sin 2a}{\sin (2a + CAD)}$$

$$= \frac{AC \sin 2a}{a \sin 2a + b \cos 2a}$$

$$= \frac{2(a \sin a + b \cos a) \cos a}{a \sin 2a + b \cos 2a}.$$

$$\therefore \frac{FG}{DE} = 2 \frac{(a \sin a + b \cos a)^2 \cos a}{(a \sin 2a + b \cos 2a)(a \sin 2a + b \cos 2a - c \sin a)}$$

$$\text{or } FG = 2c \frac{(a \sin a + b \cos a)^2 \cos a}{(a \sin 2a + b \cos 2a)(a \sin 2a + b \cos 2a - c \sin a)}.$$

241. A speck is in the middle of the back of an isosceles prism of glass; to find the angle subtended by the two images of the speck seen by an eye close to the edge of the prism.

Let ABD (fig. 103) be a section of the prism by a plane through the speck C perpendicular to its edge A . Draw An perpendicular to AD , and at the point A make the angle nAq such that $\sin nAq = \mu \sin nAC$, μ being the refractive index of the prism. Then Aq is the direction in which the image of C which is formed by refraction at AD is seen by the eye at A . If $q'AC = qAC$, Aq' is the direction in which the other image is seen, and $qAq' = \theta$ is the angular distance of the two images.

$$\text{If } DAB = \iota, \quad CAn = \frac{\pi}{2} - CAD = \frac{\pi}{2} - \frac{\iota}{2}.$$

$$\therefore \sin \left(\frac{\pi}{2} - \frac{\iota}{2} + \frac{\theta}{2} \right) = \mu \sin \left(\frac{\pi}{2} - \frac{\iota}{2} \right),$$

$$\cos \left(\frac{\iota}{2} - \frac{\theta}{2} \right) = \mu \cos \frac{\iota}{2},$$

$$\theta = \iota - 2 \cos^{-1} \left(\mu \cos \frac{\iota}{2} \right).$$

OBS. The images will not be formed in the manner supposed if CAn exceed $\sin^{-1} \frac{1}{\mu}$ the critical angle of the medium, i. e. if $\frac{\pi}{2} - \frac{\iota}{2}$ be $> \sin^{-1} \frac{1}{\mu}$, or $\cos \frac{\iota}{2} > \frac{1}{\mu}$.

242. A luminous circular ring is placed before a prism, and is viewed by an eye close to the edge of the prism whose axis passes through the center of the ring and is perpendicular to its plane: to find the nature of the visible image when it is seen most distinctly, the surface of the prism on which light is first incident being parallel to the plane of the ring.

Let A (fig. 104) be the edge of the prism, AN_1 AN_2 normals to the first and the second surfaces, the center of the ring lying in the former line which is also the axis of the eye. If the image of the ring be distinct, an extension of the investigation of (84) shews that the axis of the pencil by which any point of it is seen has the same inclination to AN_2 at emergence from the prism as it has to AN_1 at incidence. The plane of the ring being perpendicular to AN_1 and its center in the same line, the latter angle is the same for all points of the ring; the former angle therefore is also constant, or the axes of the pencils by which the image is seen lie in a conical surface of which AN_2 is the axis. The image being seen projected on a plane perpendicular to AN_1 is an ellipse.

If $N_1AN_2 = \beta$, and if α be the inclination of the axis of each pencil to AN_1 before incidence on the prism, and to AN_2 at emergence, the ratio of the minor to the major axis of the ellipse =
$$\frac{\sqrt{\cos(\alpha + \beta) \cos(\alpha - \beta)}}{\cos \alpha}.$$

(*Hymers' Conic Sect.* 214).

243. A wafer is viewed through a convex lens of 8 inches focal length, placed halfway between it and the eye; to find the diameter of the lens when the whole is seen, the diameter of the wafer being half an inch, and its distance from the eye 8 inches.

Let PQR (fig. 113) be the wafer, BCD the lens and E the eye, p, r in CP, CR produced the points in pqr the virtual image of the wafer which is seen by the eye corresponding to P, Q in the object. If the whole of it be just seen EBp is a straight line.

$$CE = CQ = 4 \text{ inches, } PQ = \frac{1}{4} \text{ inch,}$$

$$\frac{1}{Cq} - \frac{1}{CQ} = -\frac{1}{8} \quad (87), \quad \therefore Cq = 8 \text{ inches, } Eq = 12 \text{ inches.}$$

$$\frac{BC}{pq} = \frac{EC}{Eq}, \quad \frac{pq}{PQ} = \frac{Cq}{CQ},$$

$$\therefore BC = PQ \cdot \frac{EC}{Eq} \cdot \frac{Cq}{CQ} = \frac{1}{6},$$

and the diameter of the lens = $\frac{1}{3}$ inch.

244. A compound microscope is composed of two convex lenses of focal lengths 1 and 3 inches, separated by a distance of two inches; to find the position of a small object when it is most distinctly seen, and the angle which it subtends at an eye applied to the second lens.

Let C, c (fig. 105) be the centers of the first and second lenses, PQ the object viewed, pq its image formed by pencils centrically refracted through the first lens. This image, if it be seen most distinctly, is at the principal focus of the second lens (158),

$$\therefore cq = 3 \text{ inches, } Cq = 1 \text{ inch.}$$

The focal length of the first lens being 1 inch,

$$\frac{1}{Cq} - \frac{1}{CQ} = -1, \quad CQ = \frac{1}{2} \text{ inch,}$$

which determines the position of the object.

If p be the point in the image corresponding to the point P of the object, and p, c be joined, pcq is the angle which PQ seen through the microscope subtends at the eye.

The object being supposed small,

$$\begin{aligned} pcq &= \frac{pq}{cq} \\ &= \frac{PQ}{cq} \cdot \frac{Cq}{CQ} \\ &= \frac{2}{3} PQ. \end{aligned}$$

If the least distance of distinct vision be 8 inches, the object PQ viewed by the naked eye cannot subtend a greater angle than $\frac{PQ}{8}$. The ratio of the former angle to this is $\frac{16}{3}$ or 5 nearly.

245. The ends of a glass cylinder are worked into portions of a convex and concave spherical surface having their centers in the axis of the cylinder: to find the distance of these surfaces that an eye placed at the concave surface may see the image of a distant object most distinctly, and to determine the angle which the object subtends at the eye.

Let O, O' (fig. 106) be the centers of the surfaces which their common axis meets in the points A, A' respectively, μ the refractive index of the medium. Let PQ be the object viewed, the distance of which is supposed so great that all the rays of a pencil from any point of it which fall upon the cylinder may be considered parallel, pq the image of it formed by direct refraction at the first surface, p, q being the geometrical foci of P, Q respectively.

$$\therefore \frac{\mu}{Aq} = \frac{\mu - 1}{AO} \quad (32)$$

$$\text{or } Aq = \frac{\mu \cdot AO}{\mu - 1}.$$

Now in order that vision may be most distinct, the pencil from any point of PQ must at emergence from the second surface of the cylinder consist of parallel rays, or if the course of the pencil be supposed reversed (61), a pencil of parallel rays refracted directly into the cylinder through its second surface must have q for its geometrical focus;

$$\therefore A'q = \frac{\mu \cdot A'O'}{\mu - 1}.$$

$$\therefore AA' = \frac{\mu(AO - A'O')}{\mu - 1}.$$

Again if pO' be joined, $pO'q$ is the angle which PQ seen through the cylinder subtends at the eye.

$$\text{Now } Oq = \frac{AO}{\mu - 1}, \quad O'q = \frac{A'O'}{\mu - 1}.$$

$$\begin{aligned} \therefore pO'q &= \frac{pq}{O'q} \text{ approximately} \\ &= \frac{Oq}{O'q} \cdot \frac{PQ}{OQ} \\ &= \frac{AO}{A'O'} \cdot \frac{PQ}{OQ}. \end{aligned}$$

In consequence of its distance the object PQ seen by the naked eye would subtend the angle POQ or $\frac{PQ}{OQ}$. The ratio therefore of the angles subtended by the image of PQ and by PQ is $\frac{AO}{A'O'}$.

246. The object glass of an Astronomical telescope has a focal length of 50 inches, and the focal length of each lens of the Ramsden's eye piece is 2 inches; find the position of the eye piece when adjusted for ordinary eyes and the magnifying power of the telescope.

Let C (fig. 107) be the center of the object glass, q its principal focus, E , c the centers of the field glass and eye glass of the eye piece,

$$\therefore Cq = 50 \text{ inches, } Ec = \frac{4}{3} \text{ inches (194).}$$

Let q' be the place of the image formed by the field glass which must be at the principal focus of the eye glass,

$$\therefore cq' = 2 \text{ inches, and } Eq' = \frac{2}{3} \text{ inches.}$$

Now
$$\frac{1}{Eq'} - \frac{1}{Eq} = -\frac{1}{2} \quad (87)$$

$\therefore Eq = .5 \text{ inch, } EC = 50.5 \text{ inches,}$

which assigns the position of the eye piece.

To find the magnifying power suppose the axis of an excentrical pencil after refraction through the field glass and eye glass to cut their axis in Y and Y' respectively. From the distance of the object glass this pencil may be regarded as approximately parallel to the axis of the field glass at incidence upon it,

$\therefore EY = 2 \quad (109), \quad cY = \frac{2}{3} \text{ inches,}$

$$\frac{1}{cY'} - \frac{1}{cY} = \frac{1}{2}. \quad \therefore cY' = \frac{1}{2}.$$

Hence if f be the focal length of the simple lens equivalent to the eye piece,

$$\frac{1}{f} = \frac{cY}{EY \cdot cY'} = \frac{2}{3} \quad (114)$$

and magnifying power $= \frac{100}{3} \quad (197, 174).$

The magnifying power may also be computed without recourse to the formulæ of excentrical pencils. For cp' (fig. 76) being the direction in which the image of P is seen, PQ which subtends to the naked eye the angle PCQ subtends when seen by the telescope the angle $p'cq'$.

$$\begin{aligned} \therefore \text{magnifying power} &= \frac{p'cq'}{PCQ} \\ &= \frac{p'q'}{pq} \cdot \frac{Cq}{cq'} \\ &= \frac{Eq'}{Eq} \cdot \frac{Cq}{cq'}, \quad Epp' \text{ being a straight line,} \\ &= \frac{4}{3} \cdot \frac{50}{2} \\ &= \frac{100}{3}. \end{aligned}$$

247. The focal lengths of the larger and smaller mirrors of a Gregorian telescope are 32 and 3 inches, and the distance between their principal foci is $\frac{1}{4}$ inch, the focal lengths of the lenses of the Huygenian eye piece are 3 and 1 inches; to find the position of the eye piece when adjusted for ordinary eyes and the magnifying power of the telescope.

Let C , E be the centers of the faces, and F , f the principal foci of the larger and smaller mirrors respectively (fig. 108), and let E' , c be the centers of the field glass and eye glass of the eye piece. Then $CF = 32$ inches, $Ef = 3$ inches, $Ff = \frac{1}{4}$ inch, $E'c = 2$ inches (191). Let q be the place of the virtual image formed by reflection at the smaller mirror, q' the image formed by the field glass at the principal focus of the eye glass; $\therefore E'q' = 1$ inch.

The direct pencil whose axis is CE gives the equations

$$\frac{1}{Eq} + \frac{1}{EF} = \frac{1}{3} \quad (26), \quad \frac{1}{E'q'} - \frac{1}{E'q} = \frac{1}{3} \quad (87).$$

$$\therefore \frac{1}{E'q} = 1 - \frac{1}{3} = \frac{2}{3}, \quad E'q = \frac{3}{2},$$

$$\frac{1}{Eq} = \frac{1}{3} - \frac{4}{13} = \frac{1}{39}, \quad Eq = 39.$$

$$\therefore EE' = 39 - \frac{3}{2} = 37\frac{1}{2} \text{ inches,}$$

which determines the position of the eye piece.

To find the magnifying power suppose the axis of an excentrical pencil after refraction through the field glass and eye glass to cut their axis in Y and Y' respectively. From the distance of the small mirror the pencil may be approximately regarded as parallel to the axis of the field glass at incidence on that lens,

$$\therefore EY = 3 \quad (109), \quad cY = 1,$$

$$\frac{1}{cY'} - \frac{1}{cY} = 1,$$

$$\therefore cY' = \frac{1}{2}.$$

Hence if f_e be the focal length of the simple lens equivalent to the eye piece,

$$\frac{1}{f_e} = \frac{cY}{cY' \cdot E'Y} = \frac{2}{3};$$

$$\begin{aligned} \therefore \text{magnifying power} &= \frac{2}{3} \times 32 \times \frac{39}{3} \quad (182) \\ &= 277 \text{ nearly.} \end{aligned}$$

The magnifying power might also have been obtained by a method similar to that used in the preceding example.

To find how the position of the small mirror must be altered for an eye which requires pencils to diverge from a distance of 12 inches to produce distinct vision.

The eye piece retaining the position which it has been supposed to have, $E'C$ is still $= 2\frac{1}{4}$ inches. The position of q' is now determined by the equation

$$\frac{1}{12} - \frac{1}{cq'} = -1,$$

$$\therefore cq' = \frac{12}{13}, \quad E'q' = \frac{14}{13};$$

$$\frac{1}{E'q'} - \frac{1}{E'q} = \frac{1}{3},$$

$$\therefore E'q = \frac{42}{25}, \quad Cq = 3.93;$$

$$\frac{1}{EC + 3.93} + \frac{1}{EC - 32} = \frac{1}{3},$$

$$\therefore EC = 35.249 \text{ inches,}$$

and in the former adjustment $EC = 35.25$, so that it has been necessary to move the small mirror towards the large one through .001 inches.

248. If the angular distance between the sun's center and a distant station be measured by a Hadley's sextant,

and the index moved forward through an angle equal to the angle between the axis of the telescope and a normal to the fixed mirror without moving the sextant, the sun's light will be reflected from the moveable mirror to the distant station.

From any point O (fig. 109) in the plane through the sun's center, the observed station, and the axis of the telescope, draw straight lines OS , OP through the center of the sun and the station, ON_1 , ON_2 parallel to the normals of the moveable and fixed reflector when the center of the sun's image is seen in coincidence with the station, ON' parallel to the normal of the moveable glass when the sun's light is reflected by it in direction OP .

$$\therefore SOP = 2N'OP.$$

$$\text{But } SOP = 2N_1ON_2,$$

$$\therefore N'OP = N_1ON_2,$$

and if the common part N_1OP be taken away,

$$N'ON_1 = N_2OP.$$

Now OP is the direction of the axis of the telescope, therefore $N'ON_1$ the angle through which a normal to the index glass and consequently the index must be moved, is equal to the angle between the axis of the telescope and a normal to the fixed mirror.

249. To find the correction to the angular distance of two objects observed by a sextant wherein the axis of the telescope is not exactly perpendicular to the intersection of the plane mirrors.

In fig. 32*, constructed as is described in (60), let IP , IR be nearly quadrants, and equal to the angle between the axis of the telescope and the intersection of the plane mirrors. Let θ be the reading of the instrument, or double the inclination of the mirrors (209), $PR = \theta + \delta$ the angular distance of the objects, $IR = \frac{\pi}{2} - \alpha = IP$.

Now $PIR = \theta$, (60, Cor.)

$$\therefore \cos(\theta + \delta) = \sin^2 \alpha + \cos^2 \alpha \cdot \cos \theta,$$

which gives the correct angular distance of the objects.

Also α and δ being small, we have approximately

$$\delta = -\alpha^2 \tan \frac{\theta}{2},$$

the required correction to the reading of the limb.

If n be the number of seconds in the angle α , the correction in seconds

$$= -n^2 \sin 1'' \cdot \tan \frac{\theta}{2}.$$

250. A small pencil of homogeneous light proceeding from a point Q (fig. 112) falls on a sphere in a plane through Q and C the center of the sphere. If ϕ , ϕ' be the angles of incidence and refraction of the axis of the pencil, a the radius of the sphere, u the distance of Q from P the point of incidence of the axis, then when the rays of the pencil emerge parallel after one internal reflection,

$$\frac{u}{a} = \cos \phi \cdot \frac{4 \cos \phi - \mu \cos \phi'}{2 \mu \cos \phi' - 4 \cos \phi}.$$

Let $PQC = \theta$; $\phi + \delta\phi$, $\phi' + \delta\phi'$ the angles of incidence and refraction of a contiguous ray of the pencil, $\delta\theta$ the corresponding increment of θ for this ray.

$$\text{Then } \frac{u}{a} = \frac{\sin(\phi - \theta)}{\sin \theta} = \sin \phi \cot \theta - \cos \phi;$$

also $\frac{\sin \phi}{\sin \theta} = \frac{CQ}{a}$, and is the same for all rays of the pencil.

$$\therefore \tan \phi \cdot \delta\theta = \tan \theta \cdot \delta\phi, \quad (1)$$

$$\text{and } \cos \phi \cdot \delta\phi = \mu \cos \phi' \delta\phi'. \quad (2).$$

Now the deviation of QP at emergence $= \pi + 2\phi - 4\phi'$, and if the pencil emerge in a parallel state, the difference of

deviation for two rays must be the difference of the angle PQC for the same two, or

$$2\delta\phi - 4\delta\phi' = \delta\theta. \quad (3).$$

If $\delta\phi$, $\delta\phi'$, $\delta\theta$ be eliminated between (1), (2), and (3),

$$\begin{aligned} \cot\theta \sin\phi &= \frac{\mu \cos\phi \cos\phi'}{2\mu \cos\phi' - 4\cos\phi} \\ \therefore \frac{u}{a} &= \cos\phi \cdot \frac{4\cos\phi - \mu \cos\phi'}{2\mu \cos\phi' - 4\cos\phi}. \end{aligned}$$

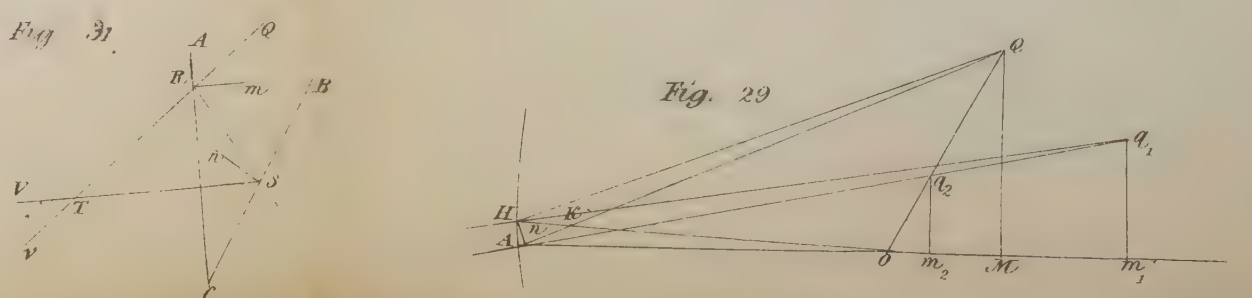
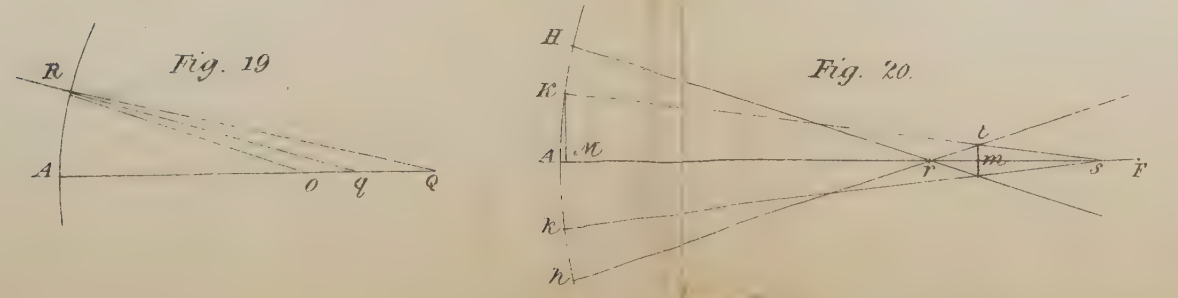
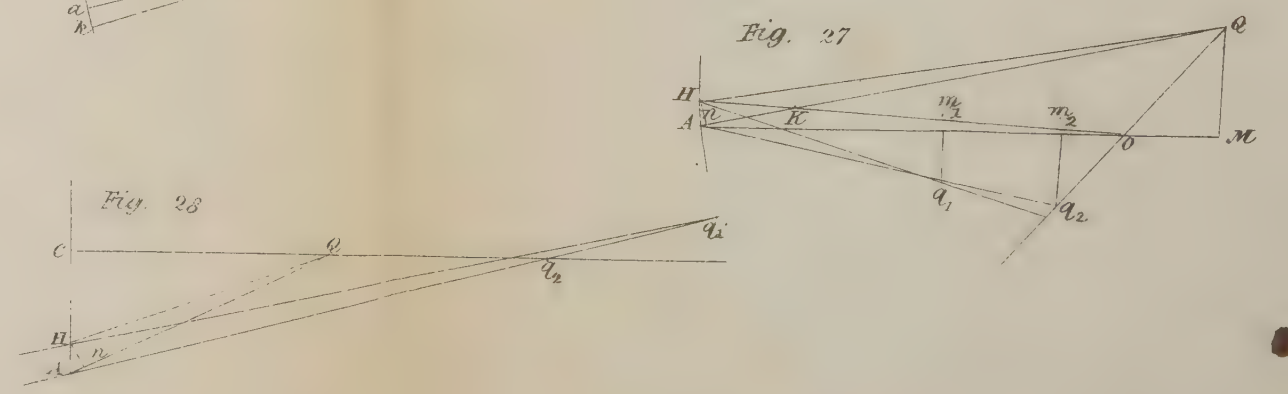
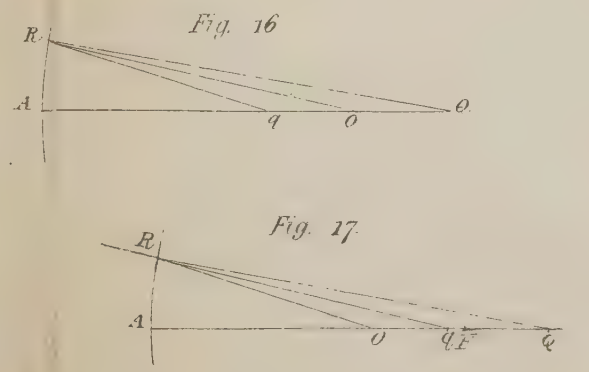
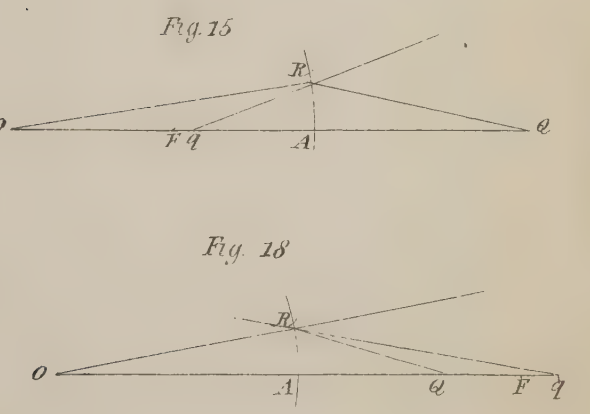
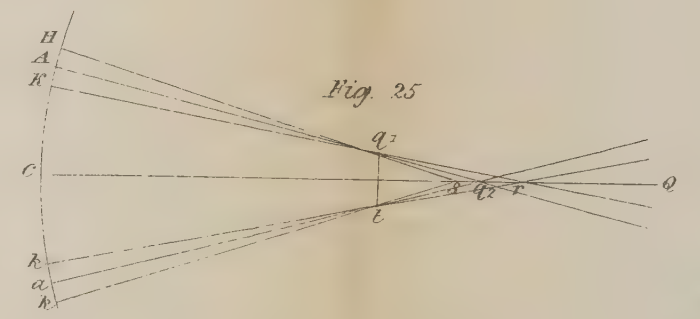
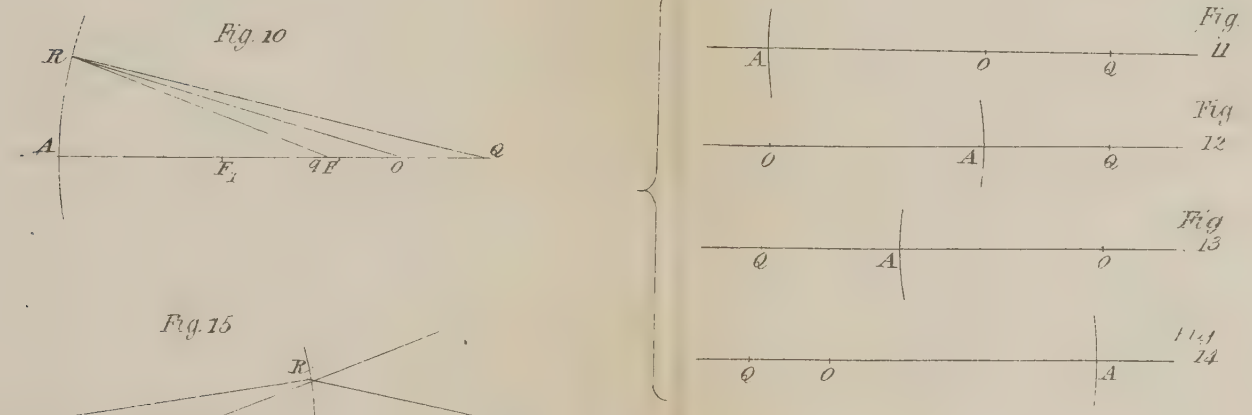
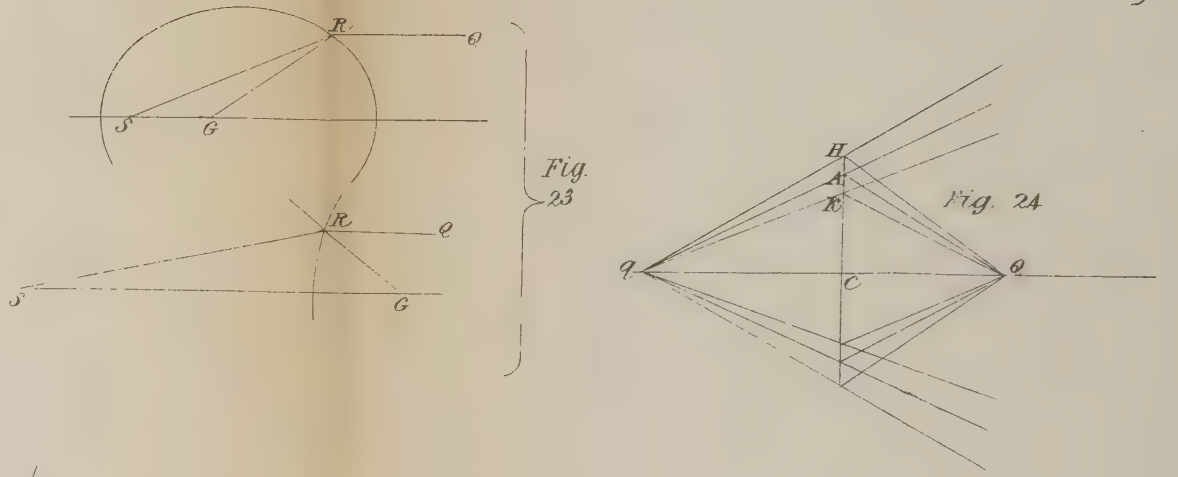
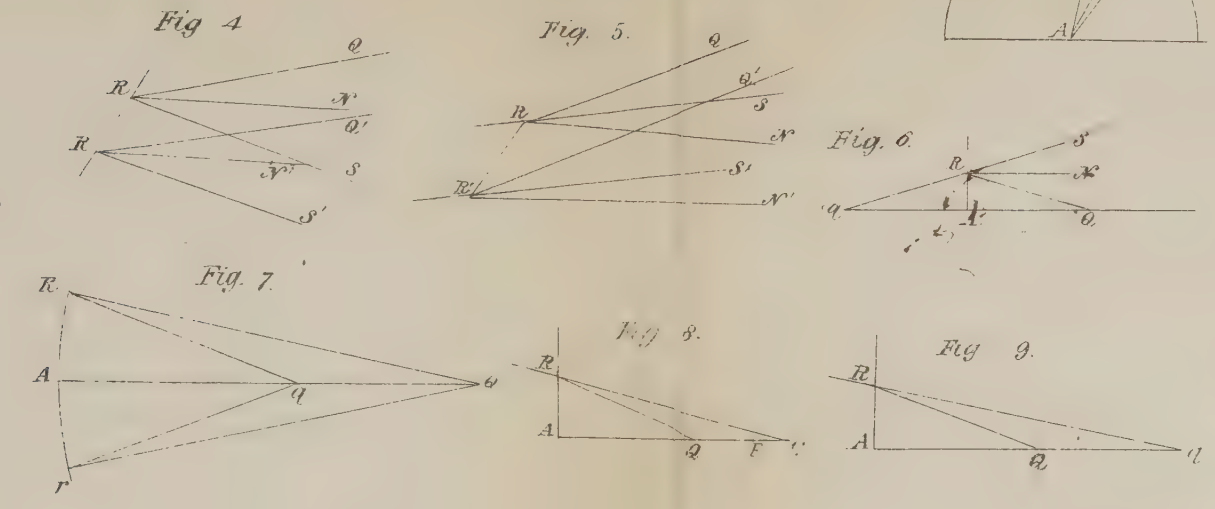
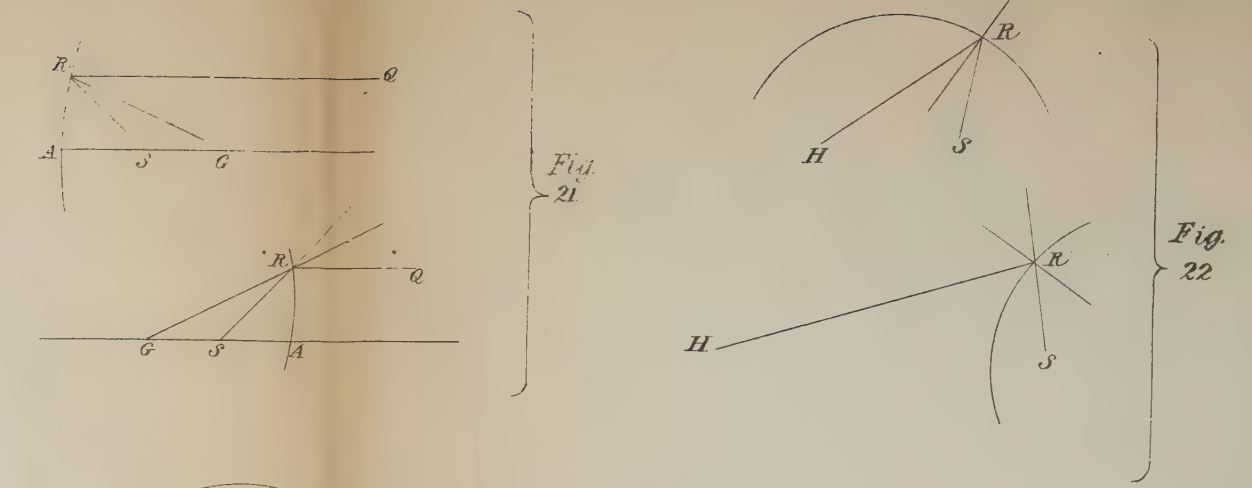
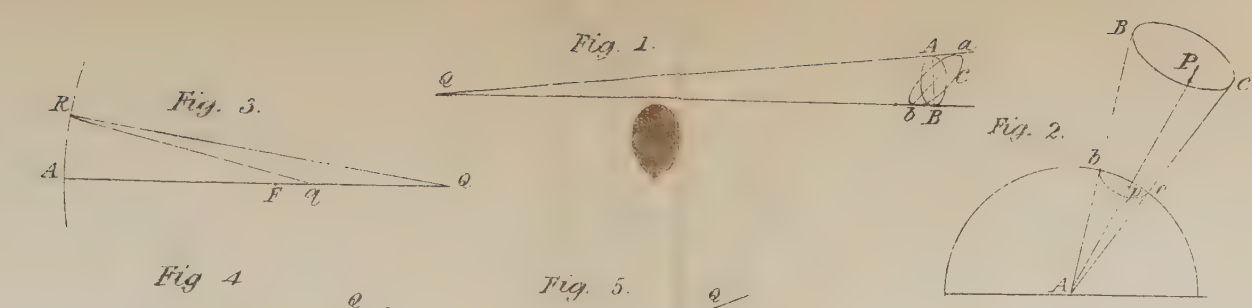
251. The position of the sun being given when a rainbow is formed, to find the point where the bow of any colour meets the horizontal plane through the eye of the observer.

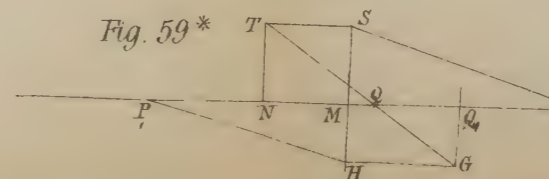
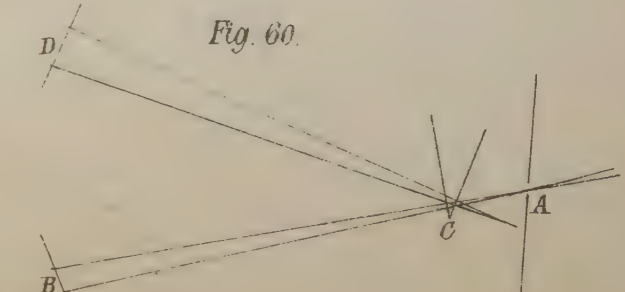
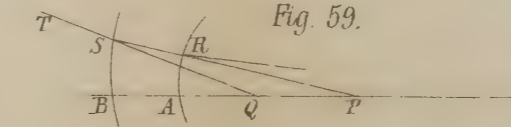
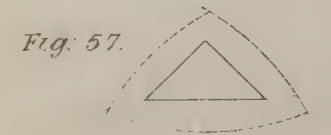
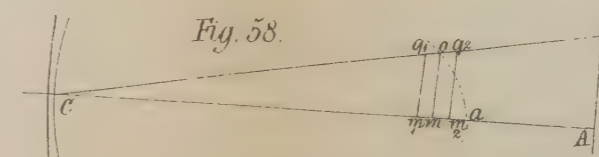
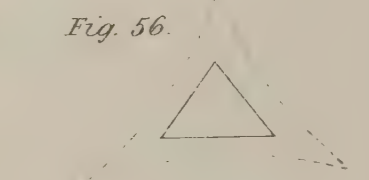
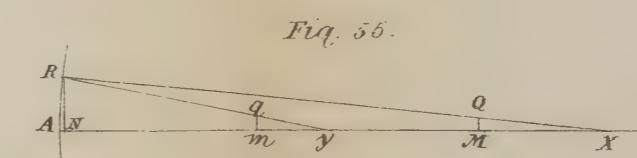
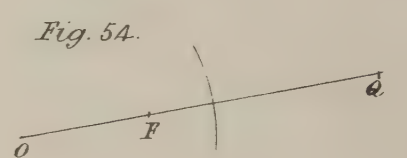
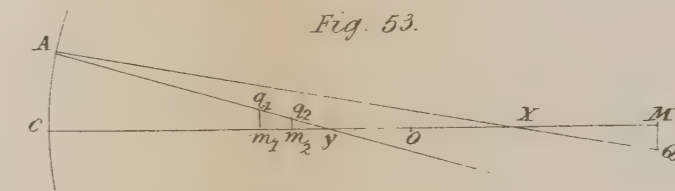
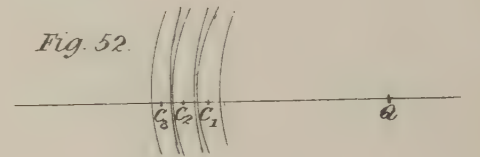
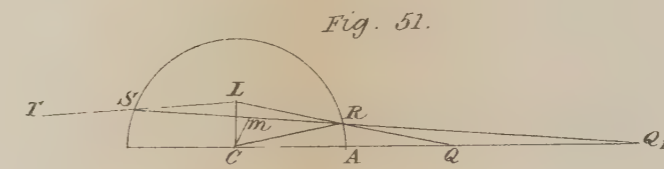
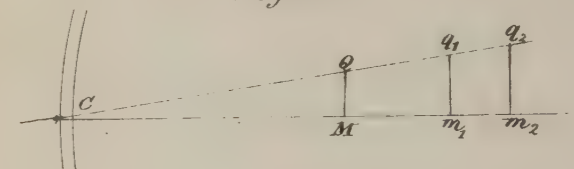
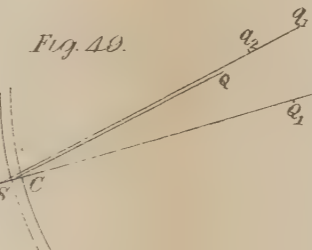
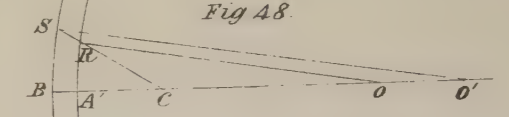
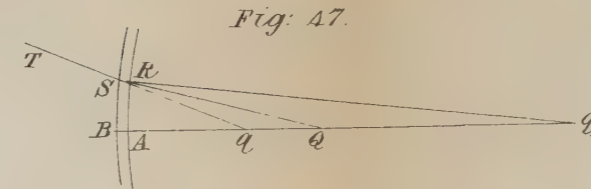
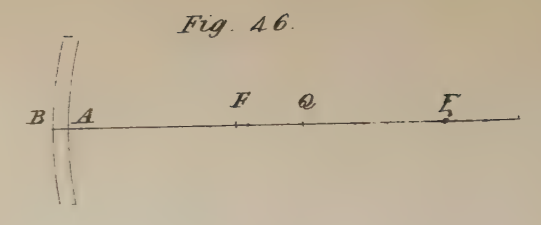
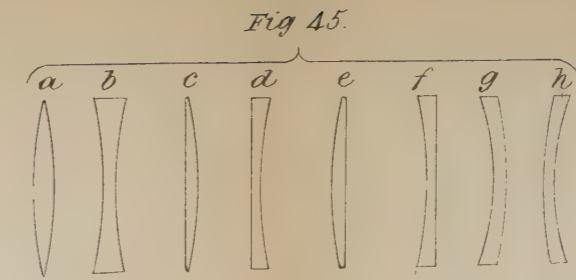
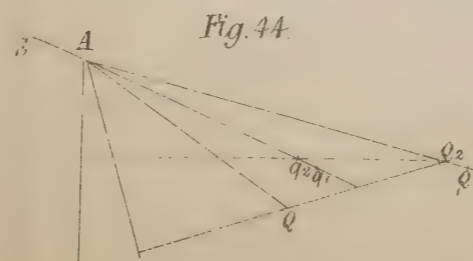
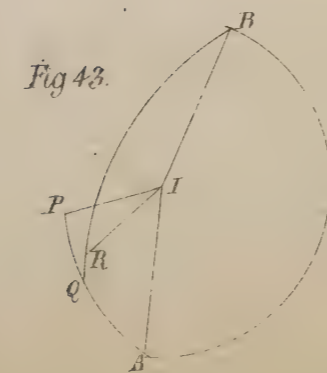
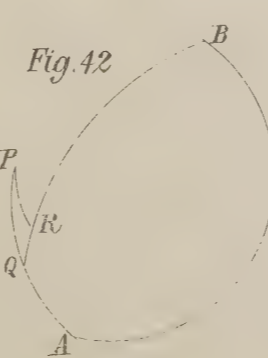
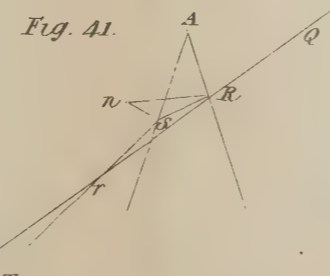
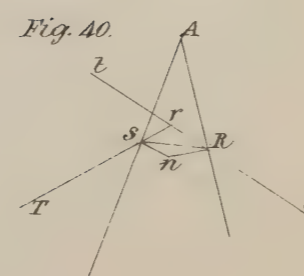
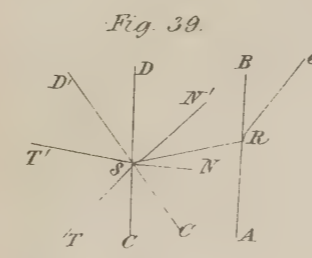
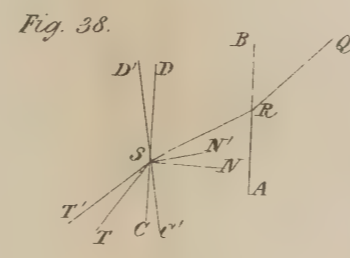
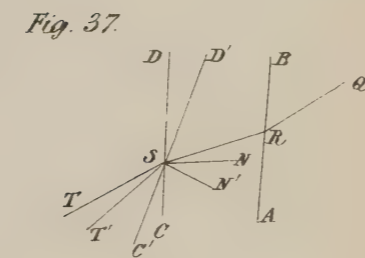
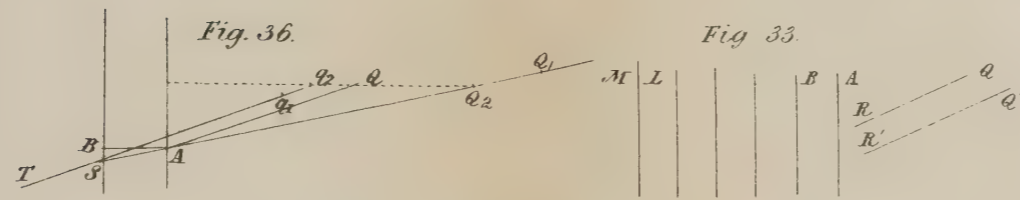
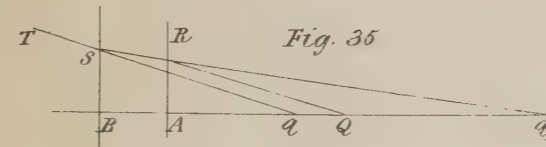
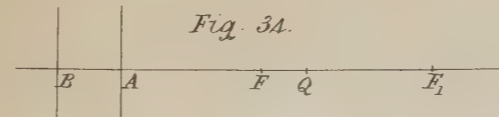
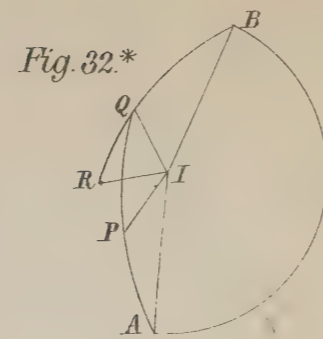
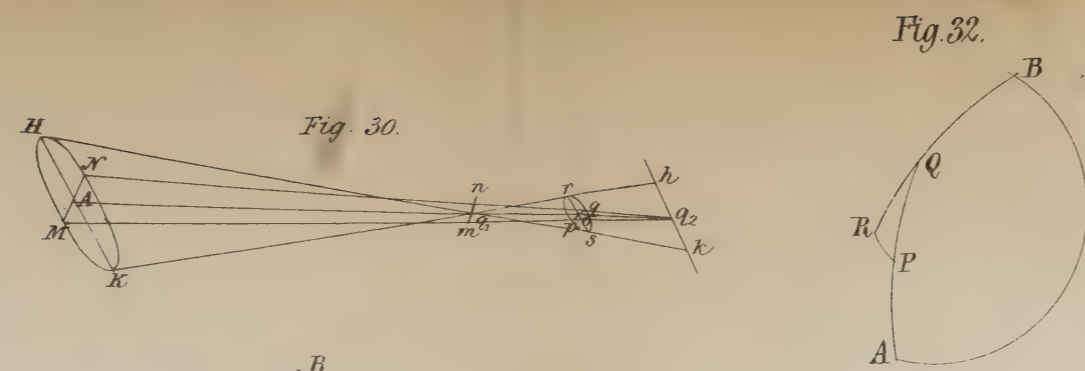
Let O (fig. 111) be the eye of the observer regarded as the center of the celestial sphere, S the sun's center; let a vertical great circle through S intersect the horizon in the straight line OA , and let PNQ the plane of the bow of a given colour referred to the celestial sphere intersect the horizon in NQ . Produce SO to meet the plane of the bow in C its center.

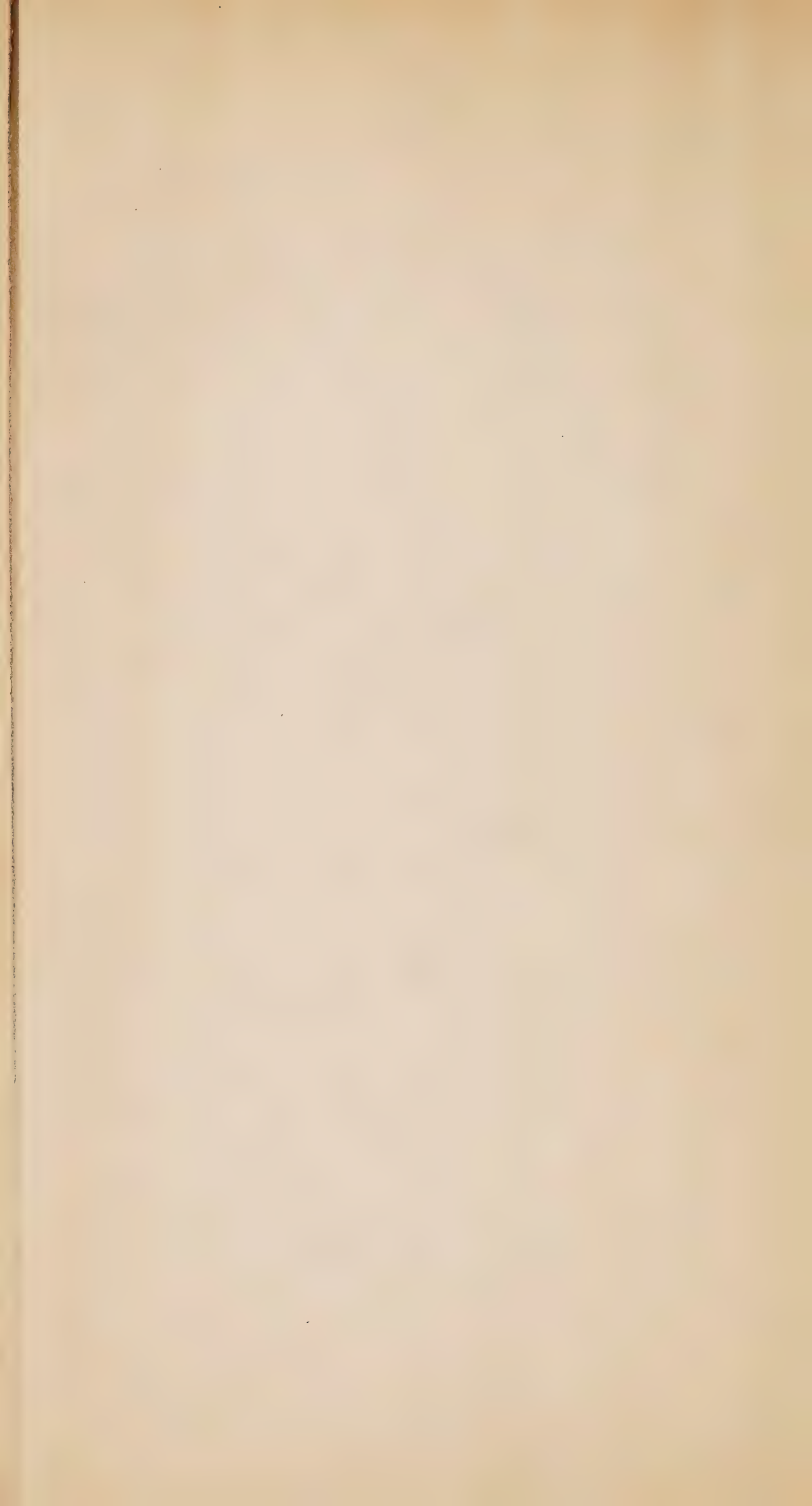
Let $COA = \alpha$ the altitude of the sun's center, $POC = \beta$ the radius of the bow for the given colour (220).

$$\begin{aligned} \therefore OQ^2 \sin^2 QOA &= QN^2 \\ &= CQ^2 - CN^2 \\ &= CP^2 - OC^2 \tan^2 \alpha \\ &= OP^2 \{ \sin^2 \beta - \cos^2 \beta \tan^2 \alpha \}; \\ \therefore \sin QOA &= \frac{\sqrt{\sin(\beta - \alpha) \cdot \sin(\beta + \alpha)}}{\cos \alpha}. \end{aligned}$$

If QOA be determined from this equation, and if A be the azimuth of the sun's center, the angular distance from the meridian of the more distant point in which the given rainbow meets the horizon $= QOA + A$.







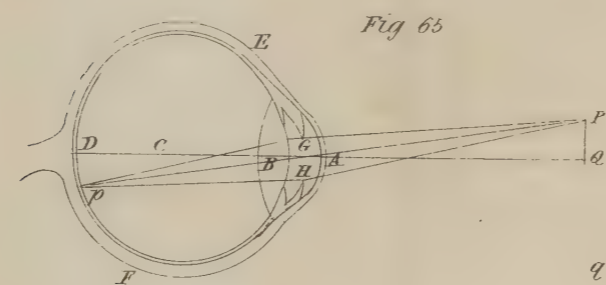
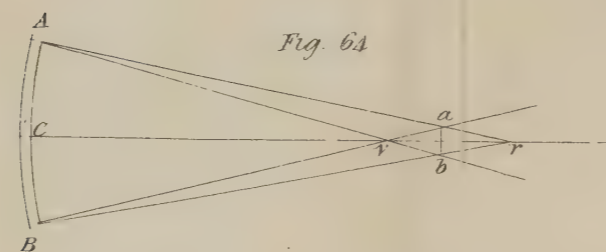
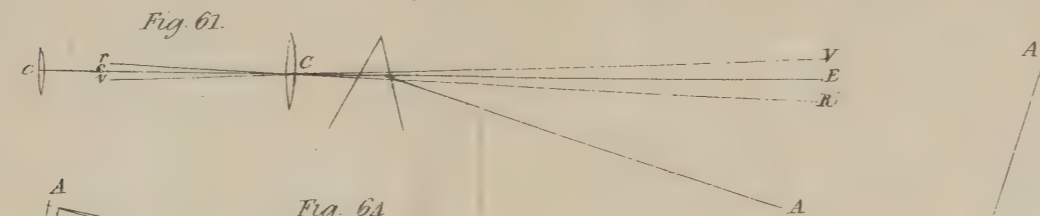
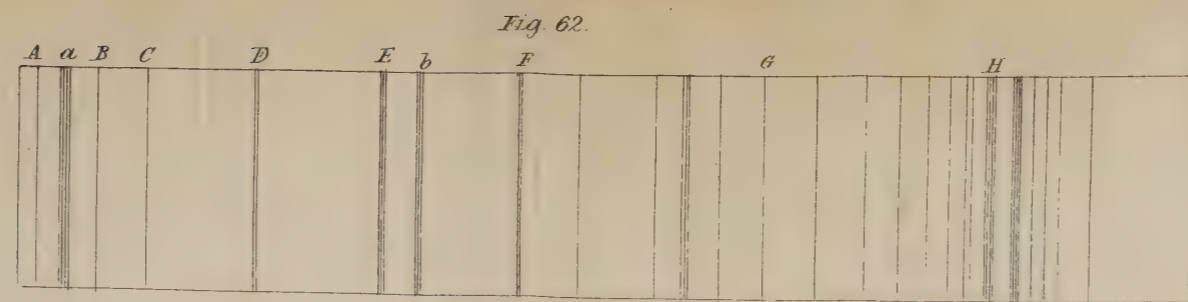


Fig. 63.

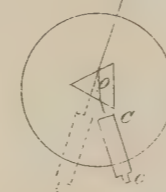


Fig. 68.

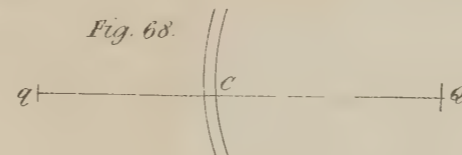


Fig. 66.



Fig. 67.

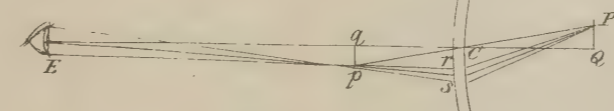


Fig. 69.

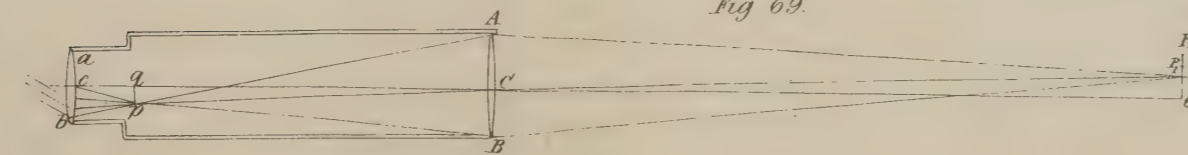


Fig. 70.

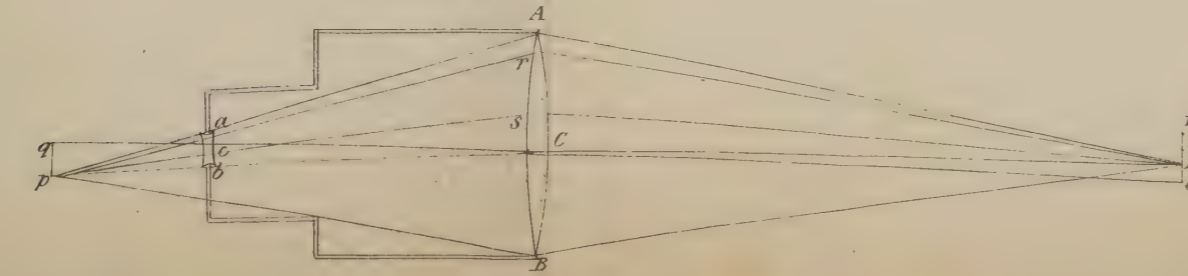


Fig. 71.

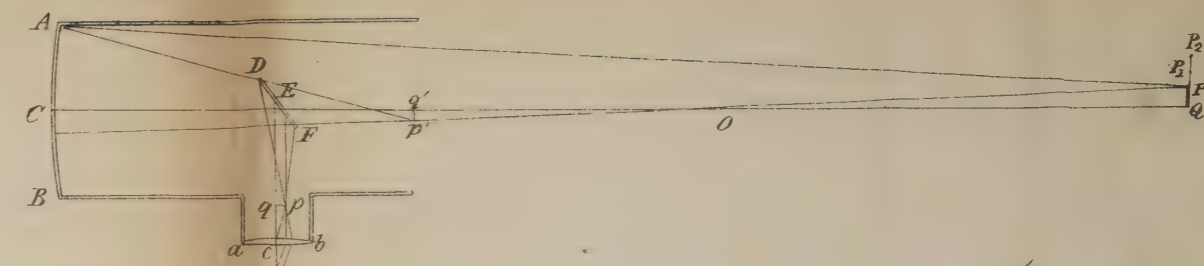


Fig. 72.



Fig. 73.

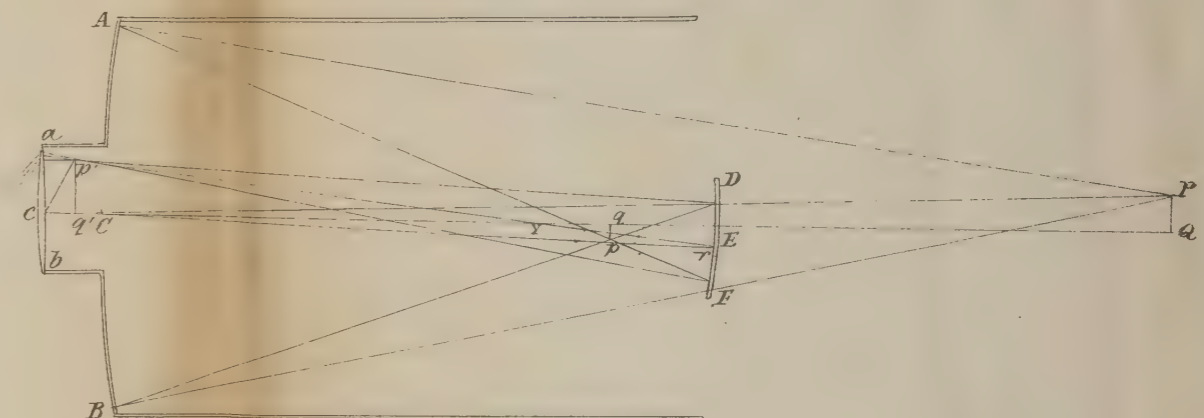


Fig. 74.

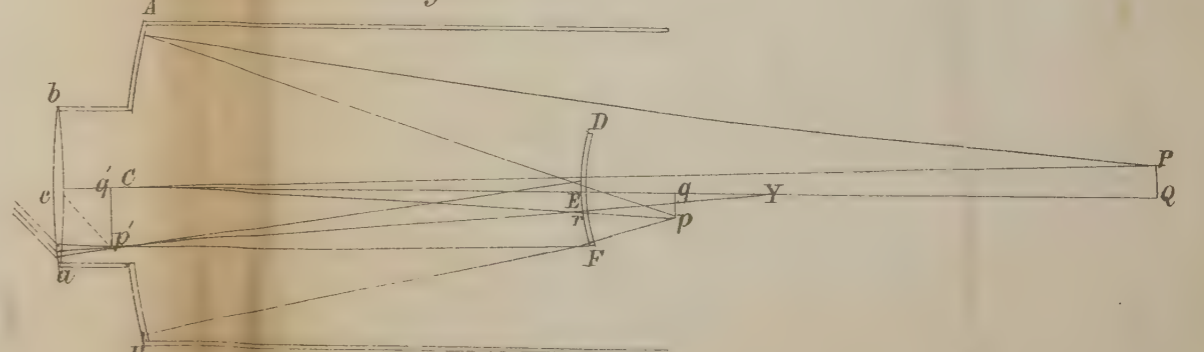
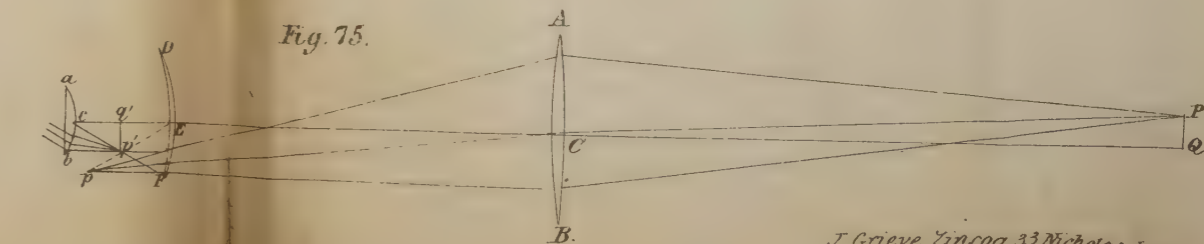
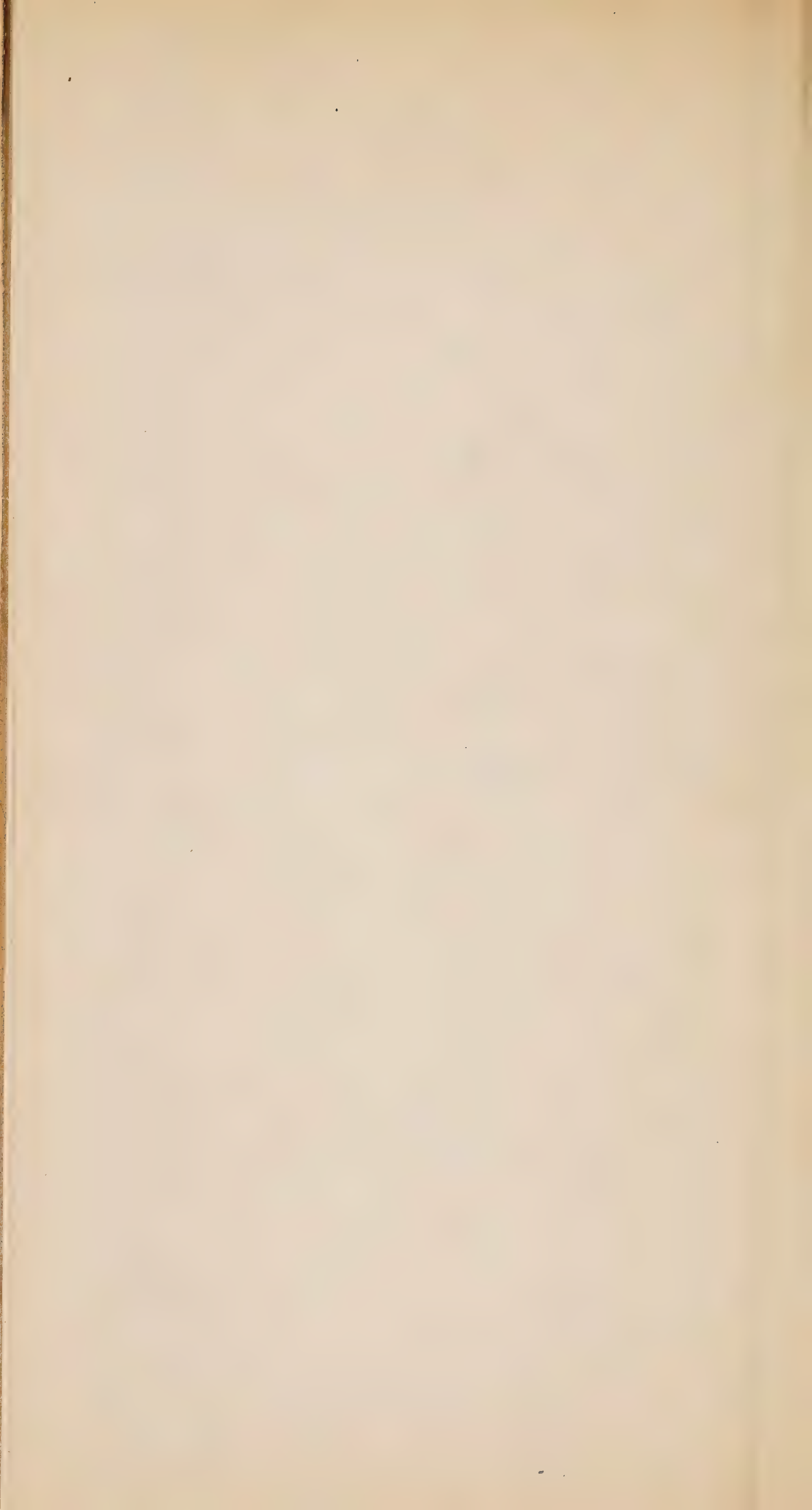


Fig. 75.





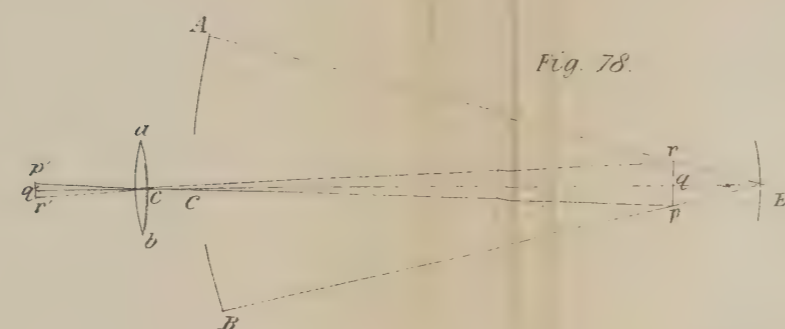
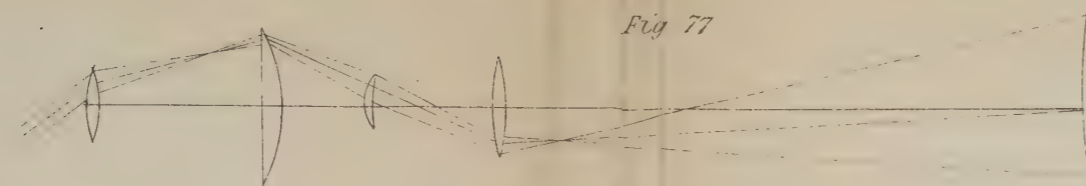
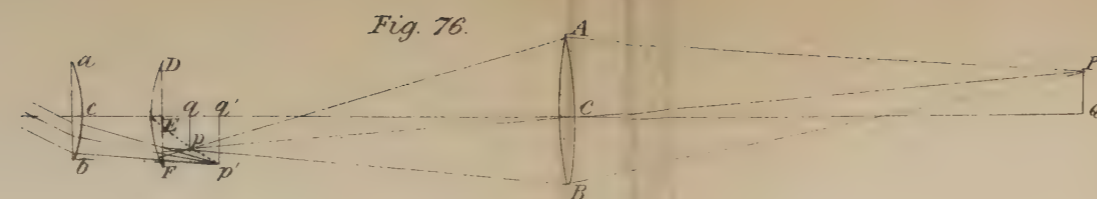


Fig. 79.

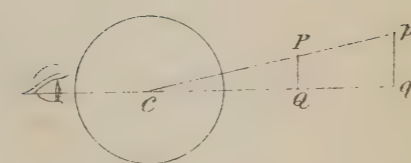


Fig. 80.

